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Larry Dean Ludeking
Iowa State University

LUDEKING, LARRY DEAN

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Multiple scattering theory and applications for intermediate energy reactions of nuclei

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Larry Dean Ludeking

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I. INTRODUCTION

A substantial variety of approaches to multiple scattering theory appears in the literature with wide variations in the physical systems and processes to which they are addressed. The unique physical situation available for study in the collision of nuclei at intermediate and high energies prompts us to consider a subset of those approaches in which there is an ordered sequestering of the degrees of freedom into groups of constituents, which either participate directly in the reaction or remain essentially passive spectators of the interactions.

The history of multiple scattering theory constitutes a field rich in formal developments. One of the earliest formally exact treatments of the three-body problem was given by Faddeev (1,2). Alternative treatments were presented in the same period by Weinberg (3,4,5) and Lovelace (6). There followed a virtual deluge of formal treatments of n-body scattering formalisms, a sample of which are included in references (7,8,9). Some of the more recent work on n-particle reaction formalism may be found in references (10,11,12,13). The approach followed in these references has been primarily directed at the development of a sound mathematical formalism for describing n-particle interactions. Unfortunately the integral structure of these approaches often presents a formidable barrier to practical calculations.

More directly applicable approaches to multiple scattering theory have developed from the work of Watson (14), Glauber (15), and Kerman, McManus and Thaler (KMT) (16). The emphasis of these references has

been elastic scattering at intermediate energies. In recent years extensive application has been made of these last three approaches to both elastic and inelastic scattering reactions.

A large amount of literature has been devoted to the investigation of corrections to these approaches. This is particularly true of the Glauber multiple scattering phenomenology, see for examples references (17-22).

Recently Ernst, Londergan, Miller and Thaler (ELMT) (23) developed a correlation expansion for the many-body optical potential. This work was supplemented by Siciliano and Thaler (ST) (24) in their treatment of a spectator expansion for multiple scattering theory. My work in Chapter II extends this development from particle-nucleus interactions to cluster-cluster interactions (25). Kowalski has recently submitted a somewhat different generalization of cluster-cluster multiple-scattering (26). His work is complementary to the approach I take in Chapter II.

I show schematically how the mean-field effects arising from many-body interactions may be accommodated. Furthermore, in outlining a reaction operator formalism I indicate how Pauli effects may be included in a natural manner. Even in the particle-nucleus limit this approach constitutes an extension of the work of ST (24).

While multi-body interactions[1] (27) (more than two or three body) are not normally addressed in nuclear applications, the formal development

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[1]Inclusion of antisymmetrization in the channel interaction (even in the case of purely two-body potentials) gives rise to effective many-body forces through exchange processes. In principle it is possible to include antisymmetrization by specification of the higher order terms in the spectator expansion (28).

allows for their presence and may be applied to a wide variety of reacting systems. In this spirit I attempt to limit the restrictive assumptions on the character (e.g., fermions or bosons) and the dynamical framework (i.e., specific nature of the interactions) governing the behavior of the constituents in the composite systems. As an example of possible applications for multi-particle forces, this development might be used to describe nucleon-nucleon scattering as the interaction of two systems of quarks.

Chapter II deals with the generalized spectator expansion. I begin with a brief review of the scattering matrix S and the transition matrix τ. Section B introduces certain notation conventions and definitions of the general Hamiltonian for two interacting clusters. I show how the Hamiltonian may be reexpressed in terms of mean fields and provide a methodology for embedding effective few-body operators in the N-body problem. In Section C the generalization of the correlation expansion of Siciliano and Thaler (24) to the cluster-cluster case is presented. While this expansion is reminiscent of the generalized cumulant expansion of Kubo (29), it has been developed from a set of algebraic identities. This correlation expansion I have called the generalized spectator expansion (GSE). Methods for reorganizing arbitrary n-body operators in terms of a systematic expansion of fewer-body operators are demonstrated. In Section D I have employed physical intuition to exploit the freedom inherent in such decompositions and thus delineate those quantities not formally specified by the Hamiltonian. Projection operators are used in Section E to promote the development of a many-

body reaction operators, whose two-body components resemble the Bruckner reaction matrix of nuclear structure calculations. In addition a generalized optical potential for interacting composite systems is presented. Finally in Section F, I consider the construction of the matrix elements for the two-body operators that arise in the spectator expansion.

In Chapter III, I develop the transition matrix for direct reactions, in which a cluster of particles may be removed from one composite system and absorbed to a second composite system. I demonstrate how to rearrange the transition matrix element into the familiar two-potential form of Gell-Mann and Goldberger (30). Motivated by freedom implicit in the spectator expansion I use the 'high energy approximation' (31) or equivalently the 'eikonal approximation' (which may be attributed originally to Moliere (32)). Various authors have used it since, though most citations have attributed the procedure to Glauber (15). Specific reduction is made to the application to protons incident upon a target nucleus, picking up a target neutron and leaving the residual target nucleus in a bound state.

Finally in Chapter IV, I discuss the sensitivities and limitations of the eikonal approach. Of particular interest is the connection that can be shown to the zero-range approximation (33,34) often made in distorted wave approximations to (p,d) and (d,p) reactions. This is shown to be related to a zeroth order truncation of the Taylor expansion of the Fourier transform of the deuteron vertex function. Comparison is made between the differential cross section computed through leading order distortions in the density with the exact vertex

function and the truncated vertex function. Systematics for a variety of incident proton energies are shown. The relative contributions of the deuteron S and D states is also displayed.

II. FORMAL DEVELOPMENT OF SCATTERING THEORY

A. Review of the S-Matrix and T-Matrix

Any discussion of many-body scattering must begin either with a time dependent or time independent framework. For most purposes it is adequate to use the time independent approach, wherein the causality restrictions of the time dependent method are subsumed in the boundary conditions applied to the stationary states of the interacting system.

A useful concept in the treatment of complex collisions is that of the 'channel'. A channel is any possible distinct mode of fragmentation and excitation of the system which may occur in the collision process. By distinct modes, I mean that states which differ only by the interchange of identical particles, are collected into a single properly symmetrized channel. For practical reasons the 'entrance' channel (denoted by α) is restricted to two-cluster states, where the clusters are labeled A and B respectively. A and B are bound states of more elementary constituents. In the entrance channel, A and B are initially noninteracting by virtue of extreme spatial separation. The 'exit' channel (denoted by β) may include multicluster states. Schematically the reaction may be written as

$$A_{\alpha 1} + A_{\alpha 2} \rightarrow \sum_i C_{\beta_i} \tag{2.1}$$

Three types of reaction are possible. Elastic collisions, in which the entrance channel and exit channel are identical, are the simplest. Slightly more complicated are the inelastic collisions

where either or both of the initial clusters is promoted to a different state of internal excitation.

$$P + {}^{16}O \rightarrow P + {}^{16}O^{*}(2^{+}) \tag{2.2}$$

Finally, there are the rearrangement channels that may involve the exchange of constituents from one cluster to the other, or fragmentation of the initial clusters.

$${}^{12}C + {}^{16}O \rightarrow 5({}^{4}He) + {}^{6}Li + {}^{2}H \tag{2.3a}$$

$${}^{2}H + {}^{176}Yb \rightarrow {}^{3}H + {}^{175}Yb \tag{2.3b}$$

Denote by H, H_α, and H_β the Hamiltonians describing the total system, the entrance channel, and the exit channel. The external channel interactions are then defined by

$$V^{\alpha} = H - H_{\alpha} \tag{2.4a}$$

$$V^{\beta} = H - H_{\beta} \quad . \tag{2.4b}$$

Associated with each of the Hamiltonians is a Green's function (propagator) constructed in the following manner.

$$G^{(\pm)} = \lim_{\eta \to 0^{+}} (E - H \pm i\eta)^{-1} \tag{2.5a}$$

$$G_{\alpha}^{(\pm)} = \lim_{\eta \to 0^{+}} (E_{\alpha} - H_{\alpha} \pm i\eta)^{-1} \tag{2.5b}$$

$$G_{\beta}^{(\pm)} = \lim_{\eta \to 0^{+}} (E_{\beta} - H_{\beta} \pm i\eta)^{-1} \tag{2.5c}$$

In the operator representation of the propagators G, the $\lim_{\eta\to 0^+}$ is understood to be the final operation in the evaluation of matrix elements. Generally the $\lim_{\eta\to 0^+}$ will be suppressed in order to enhance notational simplicity.

The propagators (G, G_γ) satisfy the expressions

$$G^{(\pm)}(E)(E - H) = 1 \quad , \tag{2.6a}$$

$$(E - H)G^{(\pm)}(E) = 1 \quad , \tag{2.6b}$$

$$G_\gamma^{(\pm)}(E_\gamma)(E_\gamma - H_\gamma) = 1 \quad , \tag{2.7a}$$

and

$$(E_\gamma - H_\gamma)G_\gamma^{(\pm)}(E_\gamma) = 1 \quad , \tag{2.7b}$$

provided that these quantities operate only upon vectors having finite norms (35,36). (This condition is guaranteed for two-cluster channels.) The (+) and (-) correspond to the propagation of 'outgoing' and 'incoming' stationary states. That is, they reflect the imposition of boundary conditions.

The eigenvectors (eigenstates) of the Hamiltonians are denoted by $\Psi_{\gamma_i}^{(\pm)}$ and Φ_{γ_i}, where the Latin subscript specifies the eigenstate and γ refers to the channel partition. The eigenequations may be written then as

$$(H - E_{\gamma_i})\Psi_{\gamma_i}^{(\pm)} = 0 \tag{2.8}$$

$$(H_\gamma - E_{\gamma_i})\Phi_{\gamma_i} = 0 \quad . \tag{2.9}$$

In order to demonstrate the connection between the eigenstates of H and H_γ, I define an eigendefect vector $\Delta^{(\pm)}_{\gamma_i}$, such that

$$\Psi^{(\pm)}_{\gamma_i} = \Phi_{\gamma_i} + \Delta^{(\pm)}_{\gamma_i} \quad . \tag{2.10a}$$

Substitution of Eq. (2.10) into Eq. (2.8) yields

$$(E_{\gamma_i} - H)\Delta^{(\pm)}_{\gamma_i} = V^\gamma \Phi_{\gamma_i} \quad .$$

Operating from the left with $G^{(\pm)}(E_{\gamma_i})$ and using the identity of Eq. (2.6a) then gives

$$\Delta^{(\pm)}_{\gamma_i} = G^{(\pm)}(E_{\gamma_i})V^\gamma \Phi_{\gamma_i} \quad .$$

Thus Eq. (2.10a) becomes

$$\Psi^{(\pm)}_{\gamma_i} = \Phi_{\gamma_i} + G^{(\pm)}(E_{\gamma_i})V^\gamma \Phi_{\gamma_i} \quad . \tag{2.10b}$$

The Møller wave operator $\Omega^{(\pm)}_\gamma$ is defined as

$$\Omega^{(\pm)}_\gamma(E_{\gamma_i}) = 1 + G^{(\pm)}(E_{\gamma_i})V^\gamma \quad , \tag{2.11}$$

which leads to another way of denoting $\Psi^{(\pm)}_{\gamma_i}$ as

$$\Psi^{(\pm)}_{\gamma_i} = \Omega^{(\pm)}_\gamma(E_{\gamma_i})\Phi_{\gamma_i} \quad . \tag{2.10c}$$

Equation (2.10c) for $\Psi^{(\pm)}_{\gamma_i}$, suggests that the outgoing state i with eigenenergy E_{γ_i} has evolved from the stationary channel state Φ_{γ_i} through the action of the wave operator $\Omega^{(+)}_\gamma$. A similar comment could be made about the ingoing state $\Psi^{(-)}_{\alpha_j}$.

A useful identity in working with operator quantities is the resolvent expression

$$a = b + a(b^{-1} - a^{-1})b \quad . \tag{2.12}$$

From Eq. (2.12) we can derive the resolvent expression for the propagators G and G_γ. (Boundary conditions (±) on the propagators are suppressed.)

$$G = G_\gamma + G_\gamma(G_\gamma^{-1} - G^{-1})G$$

$$G(E) = G_\gamma(E) + G_\gamma(E)[(E - H_\gamma) - (E - H)]G(E)$$

and hence

$$G(E) = G_\gamma(E) + G_\gamma(Z)V^\gamma G(E) \tag{2.13a}$$

or equally

$$G(E) = G_\gamma(E) + G(E)V^\gamma G_\gamma(E) \quad . \tag{2.13b}$$

Using the Møller wave operator leads to still another representation of the propagator as

$$G(Z) = \Omega_\gamma(Z)G_\gamma(Z) \quad . \tag{2.13c}$$

The scattering matrix $S_{\alpha_i\beta_j}$ (37) is defined as the overlap of the outgoing eigenstate of H from the entrance channel with the incoming eigenstate of H into the exit channel, i.e,

$$S_{\alpha_i \to \beta_j} = \langle \psi^{(-)}_{\beta_j} | \psi^{(+)}_{\alpha_i} \rangle \quad . \tag{2.14}$$

This may be simplified by considering the difference between the eigenvectors $\Psi^{(+)}_{\alpha_i}$ and $\Psi^{(-)}_{\alpha_i}$.

$$\Psi^{(+)}_{\alpha_i} - \Psi^{(-)}_{\alpha_i} = [G^{(+)}(E_{\alpha_i}) - G^{(-)}(E_{\alpha_i})]V^{\alpha}\Phi_{\alpha_i}$$

$$= \lim_{\eta \to 0^+}\left(\frac{1}{E_{\alpha_i} - H + i\eta} - \frac{1}{E_{\alpha_i} - H - i\eta} \right)V^{\alpha}\Phi_{\alpha_i}$$

$$= - 2\pi i\, \delta(E_{\alpha_i} - H)V^{\alpha}\Phi_{\alpha_i} \quad .$$

This yields

$$\Psi^{(+)}_{\alpha_i} = \Psi^{(-)}_{\alpha_i} - 2\pi i\, \delta(E_{\alpha_i} - H)V^{\alpha}\Phi_{\alpha_i} \quad , \tag{2.15}$$

which upon substitution into Eq. (2.14) generates

$$S_{\alpha_i\beta_j} = \langle\Psi^{(-)}_{\beta_j}|\Psi^{(-)}_{\alpha_i}\rangle - 2\pi i\langle\Psi^{(-)}_{\beta_j}|\delta(E_{\alpha_i} - H)V^{\alpha}|\Phi_{\alpha_i}\rangle$$

$$= \delta_{\beta_j,\alpha_i} - 2\pi i\, \delta(E_{\alpha_i} - E_{\beta_j})\langle\Psi^{(-)}_{\beta}|V^{\alpha}|\Phi_{\alpha_i}\rangle \quad . \tag{2.16a}$$

Alternatively, beginning with $\Psi^{(-)}_{\beta}$ yields

$$S_{\alpha_i\beta_j} = \delta_{\beta_j,\alpha_i} - 2\pi i\, \delta(E_{\alpha_i} - E_{\beta_j})\langle\Phi_{\beta_j}|V^{\beta}|\Psi^{(+)}_{\alpha_i}\rangle \quad . \tag{2.16b}$$

The transition matrix $\tau_{\alpha_i\beta_j}$ (37) is defined by the expression

$$S_{\alpha_i\beta_j} = \delta_{\beta_j\alpha_i} - 2\pi i\, \delta(E_{\alpha_i} - E_{\beta_j})\tau_{\alpha_i\beta_j} \quad . \tag{2.17}$$

Then from Eqs. (2.16) and (2.17) the transition matrix amplitude is

$$\tau_{\alpha_i \to \beta_j}(E_{\alpha_i}) = \langle \Phi_{\beta_j} | V^{\beta} | \Psi^{(+)}_{\alpha_i} \rangle \tag{2.18a}$$

$$\tau_{\alpha_i \to \beta_j}(E_{\beta_j}) = \langle \Psi^{(-)}_{\beta_j} | V^{\alpha} | \Phi_{\alpha_i} \rangle \tag{2.18b}$$

Equation (2.18a) is called the 'post' form of the transition matrix since the exit channel interaction appears in the bracket; analogously, Eq. (2.18b) is denoted the 'prior' form of the transition matrix. Furthermore, Eq. (2.17) implies that Eqs. (2.18a) and (2.18b) have identical values on the energy shell ($E_{\alpha_i} = E_{\beta_j}$); however, off the energy shell ($E_{\alpha_i} \neq E_{\beta_j}$) this will not in general be true. Having obtained the transition matrix amplitude on the energy shell, the scattering cross section (37) for two-cluster to two-cluster reactions may be written as

$$\frac{d\sigma_{\alpha_i \to \beta_j}}{d\Omega_{\beta}} = \frac{\mu_{\alpha}\, \mu_{\beta}}{(2\pi\hbar^2)^2} \frac{k_{\beta_j}}{k_{\alpha_i}} |\tau_{\alpha_i \to \beta_j}|^2 \quad . \tag{2.19}$$

Here $\mu_{\alpha}(\mu_{\beta})$ is the reduced mass of the two-cluster channel $\alpha(\beta)$, and $k_{\alpha_i}(k_{\beta_j})$ is the relative momentum of the two-cluster in channel state $\alpha_i(\beta_j)$.

The transition matrix $\tau_{\alpha\beta}$ leads to the defining of a transition operator $T^{(\pm)}_{\alpha\beta}$, such that on the energy shell

$$\tau_{\alpha_i \to \beta_j} = \langle \Phi_{\beta_j} | T^{(+)}_{\alpha\beta} | \Phi_{\alpha_i} \rangle = \langle \Phi_{\beta_j} | T^{(-)\dagger}_{\alpha\beta} | \Phi_{\alpha_i} \rangle \tag{2.20}$$

Then from Eqs. (2.18) and (2.20), $T^{(\pm)}_{\alpha\beta}$ can be defined as

$$T^{(+)}_{\alpha\beta}(Z) = V^{\beta} + V^{\beta} G^{(+)}(Z) V^{\alpha} \tag{2.21a}$$

and

$$T^{(-)}_{\alpha\beta}(Z) = V^{\alpha} + V^{\beta} G^{(-)}(Z) V^{\alpha} \quad . \tag{2.22a}$$

And by using the definition of the Møller wave operator, the transition operator may be written as

$$T^{(+)}_{\alpha\beta}(Z) = V^{\beta} \Omega^{(+)}_{\alpha}(Z) \tag{2.21b}$$

and

$$T^{(-)}_{\alpha\beta}(Z) = \Omega^{(-)}_{\beta}(Z) V^{\alpha} \quad . \tag{2.22b}$$

For elastic and inelastic collisions, the entrance channel and exit channel interactions are equal; i.e., $V^{\alpha} = V^{\beta} = V$, and hence the transition operator becomes

$$T^{(\pm)}(Z) = V + V G^{(\pm)}(Z) V \quad . \tag{2.23}$$

Throughout the remainder of the formal development, the explicit (±) and (Z)-dependence shall be suppressed except where needed for clarification.

B. Hamiltonian Framework and Definitions

This section is devoted to a concise description of a general N-body Hamiltonian, partitioned into two clusters, A and B, such that

A + B = N. No restrictions are imposed upon the nature of the fundamental objects which comprise N. They are elementary in the sense that it is possible to write many-body potentials which describe their mutual interactions, but they need not all be identical.

Auxiliary potentials are introduced since in many applications important physical effects can be included in this way, even in low order expansions. No requirement is made that such auxiliary potentials be used nor that they approximate a particular form. Restrictions such as they must satisfy mean-field equations, may be imposed to facilitate rapid convergence in specific applications.

Consider an N-body system of "elementary" particles, partitioned into two clusters A and B. The Hamiltonian for the cluster A is written as

$$H^A = \sum_{i=1}^{A} (\hat{k}^i + \hat{w}^i) + \sum_{i<j}^{A} \hat{\upsilon}^{ij} + \sum_{i<j<k}^{A} \hat{\upsilon}^{ijk} + \cdots \hat{\upsilon}^{12\cdots A} , \quad (2.24)$$

where the particles in cluster A are labeled sequentially as 1, 2, 3, $\cdots$ A. Furthermore, $\hat{\upsilon}^{(\nu)}$ is defined as a multi-body interaction, where $(\nu) \subset (A)$ and $|\nu|$ is the number of particles in the subset (ν). Implicit, is the restriction that if one of the particles in the subset is removed from the system, then $\hat{\upsilon}^{(\nu)} \to 0$. As an example, let $(\nu) = 149$, then $|\nu| = 3$ and $\hat{\upsilon}^{(\nu)} = \hat{\upsilon}^{149}$ is the 3-body interaction between the objects in A labeled 1, 4, and 9.

I wish to consider the possibility of including auxiliary potentials in the formalism. To be specific, I introduce a set of auxiliary potentials that satisfy mean-field equations, however, such restrictions are not essential to the developments which follow.

A set of n-body mean-fields is defined by the averaging of the sum of the m-body interactions for $m > n$. Thus I can write schematically in terms of the ν-body interactions $\hat{v}^{(\nu)}$,

$$U^{(\nu)}(n) \equiv \sum_{(j<k<l\cdots)=(\mu)\subset(A)} \langle \hat{v}^{(\nu\mu)} \rangle_{(\mu)} \quad . \tag{2.25}$$

$$(\mu) \cap (\nu) \equiv 0 \quad , \quad |\nu| + |\mu| = n \quad , \quad |\mu| \geq 1$$

Here $U^{(\nu)}(n)$ is the $|\nu|$-body mean-field experienced by the subset (ν) due to the n-body interactions averaged over the other $|\mu|$ distinct particles. The notation $\langle \hat{v}^{(\nu\mu)} \rangle_{(\mu)}$ means the averaging of the operator $\hat{v}^{(\nu\mu)}$ over a set of basis functions for the labeled particles contained in (μ). The selection of these basis functions may be dictated by the specific application. As an example of Eq. (2.25), consider the two-body mean-field interaction for $(\nu) = 13$ when $A = 5$; then

$$U^{13}(3) = \langle \hat{v}^{123} \rangle_2 + \langle \hat{v}^{134} \rangle_4 + \langle \hat{v}^{135} \rangle_5 \quad ,$$

$$U^{13}(4) = \langle \hat{v}^{1234} \rangle_{24} + \langle \hat{v}^{1235} \rangle_{25} + \langle \hat{v}^{1345} \rangle_{45} \quad ,$$

$$U^{13}(5) = \langle \hat{v}^{12345} \rangle_{245} \quad .$$

Alternatively, it is possible to define the auxiliary potentials $U^{(\nu)}(n)$ as any 'convenient' or physically motivated set of functions, rather than obtaining them from the 'fundamental' interactions. In either choice, the total auxiliary potential of the $|\nu|$th rank becomes

$$U^{(\nu)} = \sum_{n=|\nu|+1}^{A} U^{(\nu)}(n) \quad . \tag{2.26}$$

Equation (2.24) may be rewritten in terms of the following quantities

$$h^i = \hat{k}^i + \hat{w}^i + U^i \tag{2.27}$$

$$\sum_{i<j} \upsilon^{ij} = \sum_{i<j} (\hat{\upsilon}^{ij} + U^{ij}) - \sum_{i} U^i(2) \tag{2.28a}$$

$$\sum_{i<j<k} \upsilon^{ijk} = \sum_{i<j<k} (\hat{\upsilon}^{ijk} + U^{ijk}) - \sum_{i} U^i(3) - \sum_{i<j} U^{ij}(3) \quad , \tag{2.28b}$$

and generally,

$$\sum_{(\nu)\subset (A)} \upsilon^{(\nu)} = \sum_{(\nu)\subset (A)} (\hat{\upsilon}^{(\nu)} + U^{(\nu)}) - \sum_{m=1}^{|\nu|-1} \sum_{(\mu_m)\subset (A)\ |\mu_m|=m} U^{(\mu_m)}(|\nu|) \quad . \tag{2.28c}$$

Equations (2.28a,b,c) imply that the $\upsilon^{(\nu)}$ may be defined as

$$\upsilon^{ij} \equiv \hat{\upsilon}^{ij} + U^{ij} - (\tfrac{1}{A-1}) \{U^i(2) + U^j(2)\} \tag{2.29a}$$

$$\upsilon^{ijk} \equiv \hat{\upsilon}^{ijk} + U^{ijk} - \frac{2}{(A-1)(A-2)} \{U^i(3) + U^j(3) + U^k(3)\}$$

$$-(\tfrac{1}{A-2}) \{U^{ij}(3) + U^{ik}(3) + U^{jk}(3)\} \tag{2.29b}$$

and generally

$$\upsilon^{(\nu)} \equiv \hat{\upsilon}^{(\nu)} + U^{(\nu)} - \sum_{m=2}^{|\nu|-1} [\binom{A-m}{|\nu|-m}]^{-1} \left[\sum_{(\mu_m)\subset (\nu);\ |\mu_m|=m} U^{(\mu_m)}(|\nu|) \right] \quad . \tag{2.29c}$$

With these definitions the Hamiltonian H^A can now be rewritten as

$$H^A = \sum_{i=1}^{A} h^i + \sum_{i<j}^{A} v^{ij} + \cdots v^{12} \cdots A \quad , \tag{2.30}$$

which retains the same form as the initial Hamiltonian but allows the use of auxiliary potentials.

We now introduce the subcluster interaction $V^{(\nu)}$, defined by the expression

$$V^{(\nu)} \equiv \sum_{i<j \subset (\nu)} v^{ij} + \sum_{i<j<k \subset (\nu)} v^{ijk} + \cdots \; v^{(\nu)} \quad . \tag{2.31}$$

As an example, let A = 9, $(\nu) = 127$, then $V^{127} = v^{12} + v^{17} + v^{27} + v^{127}$. It is apparent that for $(\nu) = (A)$, $V^{(A)}$ is the total interaction for the system A. That is,

$$H^A = \sum h^i + V^{(A)} \quad . \tag{2.32}$$

An analogous set of equations is obtained for the B cluster of particles by using Greek subscripts (α, β, γ, $\cdots$) in place of the Latin superscripts (i, j, k, $\cdots$) in Eqs. (2.24)-(2.31). For example, Eq. (2.30) becomes

$$H_B = \sum_{\alpha=1}^{B} h_\alpha + \sum_{\alpha>\beta}^{B} v_{\alpha\beta} + \sum_{\alpha<B<\gamma}^{B} v_{\alpha\beta\gamma} + \cdots v^{12} \cdots B \tag{2.33}$$

And we can write

$$H_B = \sum h_\alpha + V_{(B)} \quad . \tag{2.34}$$

The remainder of the equations for the cluster B are evident.

Consistent with the notation used in the preceding discussion, we define the interaction existing between clusters A and B, using Latin

(Greek) superscripts (subscripts) to refer to elements of cluster A (B).

$$V_{(B)}^{(A)} \equiv \sum_{i,\alpha}^{A,B} \hat{v}_{\alpha}^{i} + \sum_{i<j,\alpha}^{A,B} \hat{v}_{\alpha}^{ij} + \sum_{i,\alpha<\beta}^{A,B} \hat{v}_{\alpha\beta}^{i} + \cdots \hat{v}_{(B)}^{(A)} . \quad (2.35)$$

Here, $\hat{v}_{(\mu)}^{(\nu)}$ is the real $(|\nu| + |\mu|)$-body interaction, occurring between the particles (ν) of A and (μ) of B. If any one particle is removed from (ν) or (μ) then $\hat{v}_{(\mu)}^{(\nu)} \to 0$. In illustration, $\hat{v}_{14}^{2}$ is the 3-body potential between particle 2 of A and the pair 1 and 4 of B.

One could invoke a mean-field treatment of the many-body interactions between elements of A and B in a fashion similar to that describing the mean-fields for system A or B. Such a treatment could be used as a formal basis for two-center shell model reaction calculations, since the mean-fields would reflect the mutual interactions of A and B. Both nuclear and atomic applications of such calculations are common (27).

It should be clear that, having incorporated whatever mean-field effects that are desired, we could recover the form of Eq. (2.35). To signify this possibility the carets in Eq. (2.35) are deleted.

In a manner reminiscent of Eq. (2.30) a subcluster-subcluster interaction is defined by the expression

$$V_{(\mu)}^{(\nu)} \equiv \sum_{\substack{i \subset (\nu) \\ \alpha \subset (\mu)}} v_{\alpha}^{i} + \sum_{\substack{i<j, \subset (\nu) \\ \alpha \subset (\mu)}} v_{\alpha}^{ij} + \sum_{\substack{i \subset (\nu) \\ \alpha<\beta \subset (\mu)}} v_{\alpha\beta}^{i} + \cdots v_{(\mu)}^{(\nu)} \quad (2.36)$$

where $(\nu) \equiv (A)$ and $(\mu) \equiv (B)$.

In illustration of Eq. (2.36), let $(\nu) = 17$ and $(\mu) = 59$, then

$$V_{59}^{17} = v_5^1 + v_9^1 + v_5^7 + v_9^7 + v_5^{17} + v_9^{17} + v_{59}^1 + v_{59}^7 + v_{59}^{17} \quad .$$

Physically, this includes all two-body, three body, and four-body interactions which occur between the elements 1 and 7 of A and the elements 5 and 9 of B. $V_{(\mu)}^{(\nu)}$ includes all two-body through $(|\nu| + |\mu|)$-body interactions occurring between the sets (ν) and (μ), which are not purely system A or system B interactions. Specifically, $V_{(\mu)}^{(\nu)} \cap V^{(A)} \equiv 0$ and $V_{(\mu)}^{(\nu)} \cap V_{(B)} \equiv 0$.

We recognize that for $(\nu) \equiv (A)$ and $(\mu) \equiv (B)$, the total cluster-cluster interaction $V_{(B)}^{(A)}$ is recovered.

The total Hamiltonian may be written as

$$H_o = H^A + H_B \quad , \tag{2.37}$$

$$\begin{aligned} H &= H_o + V_{(B)}^{(A)} \\ &= H_o + V \quad . \end{aligned} \tag{2.38}$$

From the resolvent identity of Eqs. (2.12a,b) we obtain the connection between the propagators G and G_o, corresponding to the Hamiltonians H and H_o.

$$G = G_o + G_o V G \tag{2.39a}$$

$$G = G_o + G V G_o \tag{2.39b}$$

The transition operator T is defined by

$$T = V + VGV \quad . \tag{2.40}$$

Substituting Eq. (2.39a) into Eq. (2.40) gives

$$T = V + VG_oV + VG_oVGV$$

$$= V + VG_o(V + VGV)$$

$$T = V + VG_oT \quad . \tag{2.41a}$$

Use of Eq. (2.39b) yields

$$T = V + TG_oV \quad . \tag{2.41b}$$

One more identity may be obtained by inserting Eq. (2.39b) into Eq. (2.39a).

$$G = G_o + G_oVG$$

$$= G_o + G_oV(G_o + GVG_o)$$

$$= G_o + G_o(V + VGV)G_o$$

$$G = G_o + G_oTG_o \quad . \tag{2.42}$$

In this presentation I take the viewpoint that formal developments cast in the framework of expansions for T are desirable and present a systematic set of simplications, which offer flexibility and convenience for obtaining the transition matrix element of T. It is my contention that this framework is not only computationally feasible, but also provides insight into some phenomenological treatments.

C. Correlation Expansion of T

In the preceding sections I introduced the Hamiltonians, the notation, and general definitions. The transition operator T was shown to be expressible in terms of the full N-body Green's function G, or in terms of G_o, the channel Green's function. A practical approach to the N-body scattering problem may require some approximations. I develop expansions of T which are based upon the assumption that the scattering is dominated by a simple superposition of two-body interactions, followed by a superposition of effective three-body interactions, and so forth. Thus, in leading order, I allow pairwise encounters of the participant particles with the remaining (N - 2) particles behaving as spectators. The passive particles may, for example, be responsible for defining mean-fields in which the participants interact. In a correlational sense, the next order includes the effects of a pair of particles from one cluster and one particle from the other cluster participating with (N - 3) passive spectators, and so forth. I show how a systematic progression of such terms is related to the exact transition operator T.

I first introduce the algebraic identity of Lemma A, given in Appendix A. This Lemma is an extension of the one proven by Siciliano and Thaler (24) hereafter referred to as ST. In summary, it proves that for any many-body operator Eq. (2.43) is an identity.

$$\theta = \sum_{i=1}^{A} \sum_{\alpha=1}^{B} \phi_\alpha^i + \sum_{i<j}^{A} \sum_{\alpha=1}^{B} [\phi_\alpha^{ij} - \phi_\alpha^i - \phi_\alpha^j]$$

$$+ \sum_{i=1}^{A} \sum_{\alpha<\beta}^{B} [\phi_{\alpha\beta}^{i} - \phi_{\alpha}^{i} - \phi_{\beta}^{i}]$$

$$+ \sum_{i<j}^{A} \sum_{\alpha<\beta}^{B} [\phi_{\alpha\beta}^{ij} - \phi_{\alpha}^{ij} - \phi_{\beta}^{ij} - \phi_{\alpha\beta}^{i} - \phi_{\alpha\beta}^{j} + \phi_{\alpha}^{i} + \phi_{\alpha}^{j} + \phi_{\beta}^{i} + \phi_{\beta}^{j}]$$

$$+ \sum_{i<j<k}^{A} \sum_{\alpha=1}^{B} [\phi_{\alpha}^{ijk} - \phi_{\alpha}^{ij} - \phi_{\alpha}^{ik} - \phi_{\alpha}^{jk} + \phi_{\alpha}^{i} + \phi_{\alpha}^{j} + \phi_{\alpha}^{k}]$$

$$+ \sum_{i=1}^{A} \sum_{\alpha<\beta<\gamma}^{B} [\phi_{\alpha\beta\gamma}^{i} - \phi_{\alpha\beta}^{i} - \phi_{\alpha\gamma}^{i} - \phi_{\beta\gamma}^{i} + \phi_{\alpha}^{i} + \phi_{\beta}^{i} + \phi_{\gamma}^{i}]$$

$$+ \sum_{i<j<k}^{A} \sum_{\alpha<\beta}^{B} [\phi_{\alpha\beta}^{ijk} - \phi_{\alpha\beta}^{ij} - \phi_{\alpha\beta}^{ik} - \phi_{\alpha\beta}^{jk} - \phi_{\alpha}^{ijk} - \phi_{\beta}^{ijk}$$

$$+ \phi_{\alpha}^{ij} + \phi_{\alpha}^{ik} + \phi_{\alpha}^{jk} + \phi_{\beta}^{ij} + \phi_{\beta}^{ik} + \phi_{\beta}^{jk}$$

$$- \phi_{\alpha}^{i} - \phi_{\beta}^{i} - \phi_{\alpha}^{j} - \phi_{\beta}^{j} - \phi_{\alpha}^{k} - \phi_{\beta}^{k}]$$

$$+ \cdots$$

$$+ [\phi_{12\ldots B}^{12\ldots A\ldots}] \tag{2.43}$$

for arbitrary $\phi_{(\mu)}^{(\nu)}$, provided only that Eq. (2.44) is true

$$\theta = \phi_{(B)}^{(A)} \tag{2.44}$$

For notational convenience I introduce the quantities $\theta_{(\mu)}^{(\nu)}$ defined by

$$\theta_{\alpha}^{i} = \phi_{\alpha}^{i} \quad , \tag{2.45a}$$

$$\theta_{\alpha}^{ij} = (\phi_{\alpha}^{ij} - \phi_{\alpha}^{i} - \phi_{\alpha}^{j}) \quad , \tag{2.45b}$$

$$\theta^{i}_{\alpha\beta} = (\phi^{i}_{\alpha\beta} - \phi^{i}_{\alpha} - \phi^{i}_{\beta}) \quad , \tag{2.45c}$$

$$\theta^{ij}_{\alpha\beta} = (\phi^{ij}_{\alpha\beta} - \phi^{ij}_{\alpha} - \phi^{ij}_{\beta} - \phi^{i}_{\alpha\beta} - \phi^{j}_{\alpha\beta} + \phi^{i}_{\alpha} + \phi^{j}_{\alpha} + \phi^{i}_{\beta} + \phi^{j}_{\beta}) \quad , \tag{2.45d}$$

$$\vdots$$

$$\theta^{(A)}_{(B)} = (\phi^{(A)}_{(B)} - \cdots) \quad . \tag{2.45e}$$

Using the summation conversion introduced in Appendix A, Eq. (2.43) can be rewritten as

$$\theta = \sum \theta^{i}_{\alpha} \quad \sum \theta^{ij}_{\alpha} + \sum \theta^{i}_{\alpha\beta} + \cdots \theta^{(A)}_{(B)} \quad . \tag{2.46}$$

As pointed out by ST the arbitrariness in the identity expansion allows a great deal of flexibility in decomposing many-body operators. I exploit this freedom in developing operator expansions. The 'arbitrary' quantities will be chosen on physically motivated bases to exhibit a systematic progression in the decomposition of an N-body operator into effective 2, 3, $\cdots$ (N - 1)-body operators. Furthermore, this flexibility will be utilized to demonstrate connections between different multiple-scattering formalisms.

I expand the Green's functions G and G_o via the identity (2.43). Thus we obtain for G the expression,

$$G = \sum g^{i}_{\alpha} + \sum (g^{ij}_{\alpha} - g^{i}_{\alpha} - g^{j}_{\alpha}) + \sum (g^{i}_{\alpha\beta} - g^{i}_{\alpha} = g^{i}_{\beta})$$

$$+ \sum (g^{ij}_{\alpha\beta} - g^{ij}_{\alpha} - g^{ij}_{\beta} - g^{i}_{\alpha\beta} - g^{j}_{\alpha\beta} + g^{i}_{\alpha} + g^{j}_{\alpha} + g^{i}_{\beta} + g^{j}_{\beta})$$

$$+ \cdots \quad , \tag{2.47a}$$

$$= \sum \gamma_\alpha^i + \sum \gamma_\alpha^{ij} + \sum \gamma_{\alpha\beta}^i + \sum \gamma_{\alpha\beta}^{ij} + \cdots \quad , \tag{2.47b}$$

where the definitions of the reduced subcluster propagators (the $\gamma_{(\mu)}^{(\nu)}$'s) in terms of the subcluster propagators ($g_{(\mu)}^{(\nu)}$'s) follow from Eqs. (2.44) (2.45a-e). For G_o, we obtain an identical form, except that we now use $\tilde{g}_{(\mu)}^{(\nu)}$ and $\tilde{\gamma}_{(\mu)}^{(\nu)}$ instead of $g_{(\mu)}^{(\nu)}$ and $\gamma_{(\mu)}^{(\nu)}$.

$$G_o = \sum \tilde{g}_\alpha^i + \sum (\tilde{g}_\alpha^{ij} - \tilde{g}_\alpha^i - \tilde{g}_\alpha^j) + \cdots \quad , \tag{2.48a}$$

$$= \sum \tilde{\gamma}_\alpha^i + \sum \tilde{\gamma}_\alpha^{ij} + \cdots \quad . \tag{2.48b}$$

The only restrictions imposed on the sets $\{g_{(\mu)}^{(\nu)}\}$ and $\{\tilde{g}_{(\mu)}^{(\nu)}\}$ are that $g_{(B)}^{(A)} \equiv G$ and $\tilde{g}_{(B)}^{(A)} \equiv G_o$. These insure that Lemma A holds.

The cluster-cluster interaction $V_{(B)}^{(A)}$ may also be expanded in terms of (2.43); however, here we can immediately select the physically relevant choice for the expansion set. The subcluster-subcluster potentials $V_{(\mu)}^{(\nu)}$ defined by Eq. (2.36) satisfy the conditions of Lemma A, and thus we write

$$V_{(B)}^{(A)} = \sum V_\alpha^i + \sum (V_\alpha^{ij} - V_\alpha^i - V_\alpha^j) + \sum (V_{\alpha\beta}^i - V_\beta^i - V_\alpha^i)$$

$$+ \sum (V_{\alpha\beta}^{ij} - V_\alpha^{ij} - V_\beta^{ij} - V_{\alpha\beta}^i - V_{\alpha\beta}^j + V_\alpha^i + V_\beta^i + V_\alpha^j + V_\beta^j)$$

$$+ \cdots \quad , \tag{2.49a}$$

and

$$= \sum \upsilon_\alpha^i + \sum \upsilon_\alpha^{ij} + \sum \upsilon_{\alpha\beta}^i + \sum \upsilon_{\alpha\beta}^{ij} + \cdots \quad . \tag{2.49b}$$

It is physically apparent what each expression in Eq. (2.49a) represents, when we recall the meaning of the $v_{(\mu)}^{(\nu)}$. The element $v_{(\mu)}^{(\nu)}$ is the 'real' $|\nu| + |\mu|$-th body interaction existing between the (ν)-subcluster of A and the (μ)-subcluster of B. In the event that a particle is removed from either (ν) or (μ) to infinite spacial separation, this potential vanishes.

Appendix A includes two useful corollaries which simplify the construction of expansions for operators defined by algebraic functions of other operators. Corollary I (Addition Corollary) states that for any operator $Z = X + Y$, if the sets $\{X_{(\mu)}^{(\nu)}\}$ and $\{Y_{(\mu)}^{(\nu)}\}$ satisfy the conditions of Lemma A for X and Y respectively, then the set defined by $Z_{(\mu)}^{(\nu)} \equiv X_{(\mu)}^{(\nu)} + Y_{(\mu)}^{(\nu)}$, does so for Z. Similarly, Corollary II (Multiplication Corollary) specifies that for $Z = XY$, the set $Z_{(\mu)}^{(\nu)} \equiv X_{(\mu)}^{(\nu)}\, Y_{(\mu)}^{(\nu)}$ is a legitimate expansion set of Z.

Lemma A allows us to write the expansion of the cluster-cluster transition operator T as

$$T = \sum t_\alpha^i + \sum (t_\alpha^{ij} - t_\alpha^i - t_\alpha^j) + (t_{\alpha\beta}^i - t_\alpha^i - t_\beta^i)$$
$$+ \sum (t_{\alpha\beta}^{ij} - t_\alpha^{ij} - t_\beta^{ij} - t_{\alpha\beta}^i - t_{\alpha\beta}^j + t_\alpha^i + t_\alpha^j + t_\beta^i + t_\beta^j)$$
$$+ \cdots \quad , \tag{2.50a}$$

$$= \sum \tau_\alpha^i + \sum \tau_\alpha^{ij} + \sum \tau_{\alpha\beta}^i + \sum \tau_{\alpha\beta}^{ij} + \cdots + \tau_{(B)}^{(A)} \quad . \tag{2.50b}$$

which we will refer to as the Generalized-Spectator Expansion of the transition operator T.

Making use of the Corollaries I and II of Appendix A, the definition Eq. (2.53a) of T, Eq. (47a) for G, and Eq. (49a) for V, we obtain defining relations for the subcluster-subcluster transition operator $t^{(\nu)}_{(\mu)}$.

$$t^{i}_{\alpha} = \upsilon^{i}_{\alpha} + \upsilon^{i}_{\alpha}\, g^{i}_{\alpha}\, \upsilon^{i}_{\alpha} \tag{2.51a}$$

$$\begin{aligned} t^{ij}_{\alpha} &= (\upsilon^{ij}_{\alpha} + \upsilon^{i}_{\alpha} + \upsilon^{j}_{\alpha}) + (\upsilon^{ij}_{\alpha} + \upsilon^{i}_{\alpha} + \upsilon^{j}_{\alpha}) g^{ij}_{\alpha} (\upsilon^{ij}_{\alpha} + \upsilon^{i}_{\alpha} + \upsilon^{j}_{\alpha}) \\ &= V^{ij}_{\alpha} + V^{ij}_{\alpha}\, g^{ij}_{\alpha}\, V^{ij}_{\alpha} \end{aligned} \tag{2.51b}$$

$$t^{i}_{\alpha\beta} = V^{i}_{\alpha\beta} + V^{i}_{\alpha\beta}\, g^{i}_{\alpha\beta}\, V^{i}_{\alpha\beta} \tag{2.51c}$$

$$t^{ij}_{\alpha\beta} = V^{ij}_{\alpha\beta} + V^{ij}_{\alpha\beta}\, g^{ij}_{\alpha\beta}\, V^{ij}_{\alpha\beta} \tag{2.51d}$$

$$\vdots$$

$$t^{(\nu)}_{(\mu)} = V^{(\nu)}_{(\mu)} + V^{(\nu)}_{(\mu)}\, g^{(\nu)}_{(\mu)}\, V^{(\nu)}_{(\mu)} \tag{2.51e}$$

$$\vdots$$

$$t^{(A)}_{(B)} = V^{(A)}_{(B)} + V^{(A)}_{(B)}\, g^{(A)}_{(B)}\, V^{(A)}_{(B)} \equiv T \tag{2.51f}$$

From the form of Eqs. (2.51a-f), if is clear that for the appropriate definitions of the propagators $g^{(\nu)}_{(\mu)}$, the transition operators $t^{(\nu)}_{(\mu)}$ become purely $(|\nu| + |\mu|)$-body operators. Specifically, only the labeled particles (ν) from cluster A and (μ) from cluster B may participate actively in $t^{(\nu)}_{(\mu)}$. The remaining particles play the role of

'passive' spectators. Inspection of the terms in Eq. (2.50b) displays the symmetry under interchange of clusters A and B. In τ_α^i we note that the labeling expressly separates the i-th and α-th particles from the (A - i) + (B - α) remaining particles, and treats them as 'effective' two-body operators. The so-called passive spectators may participate to the extent of defining background mean-fields.

If, instead of using Eq. (2.47a) for G, we use Eq. (2.48a) for G_o and the iterative definition Eq. (2.53b) of T, the resulting expansion (written with $\tilde{t}_{(\mu)}^{(\nu)}$ and $\tilde{\tau}_{(\mu)}^{(\nu)}$ of Eqs. (2.50a,b) has the following definitions for the $\tilde{t}_{(\mu)}^{(\nu)}$.

$$\tilde{t}_\alpha^i = v_\alpha^i + v_\alpha^i \tilde{g}_\alpha^i \tilde{t}_\alpha^i \quad , \tag{2.52a}$$

$$\tilde{t}_\alpha^{ij} = V_\alpha^{ij} + V_\alpha^{ij} \tilde{g}_\alpha^{ij} \tilde{t}_\alpha^{ij} \quad , \tag{2.52b}$$

$$\tilde{t}_{\alpha\beta}^i = V_{\alpha\beta}^i + V_{\alpha\beta}^i \tilde{g}_{\alpha\beta}^i \tilde{t}_{\alpha\beta}^i \quad , \tag{2.52c}$$

$$\vdots$$

$$\tilde{t}_{(\mu)}^{(\nu)} = V_{(\mu)}^{(\nu)} + V_{(\mu)}^{(\nu)} \tilde{g}_{(\mu)}^{(\nu)} \tilde{t}_{(\mu)}^{(\nu)} \quad , \tag{2.52d}$$

$$\vdots$$

$$\tilde{t}_{(B)}^{(A)} = V_{(B)}^{(A)} + V_{(B)}^{(A)} \tilde{g}_{(B)}^{(A)} \tilde{t}_{(B)}^{(A)} \equiv T \quad . \tag{2.52e}$$

In principle we need not require that $\tilde{t}_{(\mu)}^{(\nu)} \equiv t_{(\mu)}^{(\nu)}$, for general (ν) and (μ). However, if this restriction is imposed, then we can show

that $\tilde{g}^{(\nu)}_{(\mu)}$ and $g^{(\nu)}_{(\mu)}$ are related by the resolvent expressions

$$g^{(\nu)}_{(\mu)} = \tilde{g}^{(\nu)}_{(\mu)} + \tilde{g}^{(\nu)}_{(\mu)} V^{(\nu)}_{(\mu)} g^{(\nu)}_{(\mu)} \tag{2.53a}$$

and

$$g^{(\nu)}_{(\mu)} = \tilde{g}^{(\nu)}_{(\mu)} + \tilde{g}^{(\nu)}_{(\mu)} t^{(\nu)}_{(\mu)} \tilde{g}^{(\nu)}_{(\mu)} \quad , \tag{2.53b}$$

which are the analogues of Eqs (2.39b) and (2.42). Figures 1 and 2 demonstrate the diagrammatic realization of these operator expressions for the transition operators $t^{(\nu)}_{(\mu)}$ and the propagators $g^{(\nu)}_{(\mu)}$ and $\tilde{g}^{(\nu)}_{(\mu)}$.

As we have seen, the Identity Lemma and Corollaries I and II have provided us with considerable simplicity in constructing operator expansions. This facilitates the establishment of decomposition sets for operators defined in terms of other operators having previously defined expansion sets. In particular the freedom inherent in Lemma A has been exploited in the choice of the subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$ or $g^{(\nu)}_{(\mu)}$ without introducing any approximations in T. A primary emphasis of this work is to introduce certain physically motivated choices for these subcluster propagators and discuss their significa .

We may associate the expressions of Eqs. (2.52a-e) with a grouping in terms of the particle rank (total number of participant particles). Thus

$$T(2) = \sum \tau^{i}_{\alpha} \quad , \tag{2.54a}$$

$$T(3) = \sum \tau^{ij}_{\alpha} + \sum \tau^{i}_{\alpha\beta} \quad , \tag{2.54b}$$

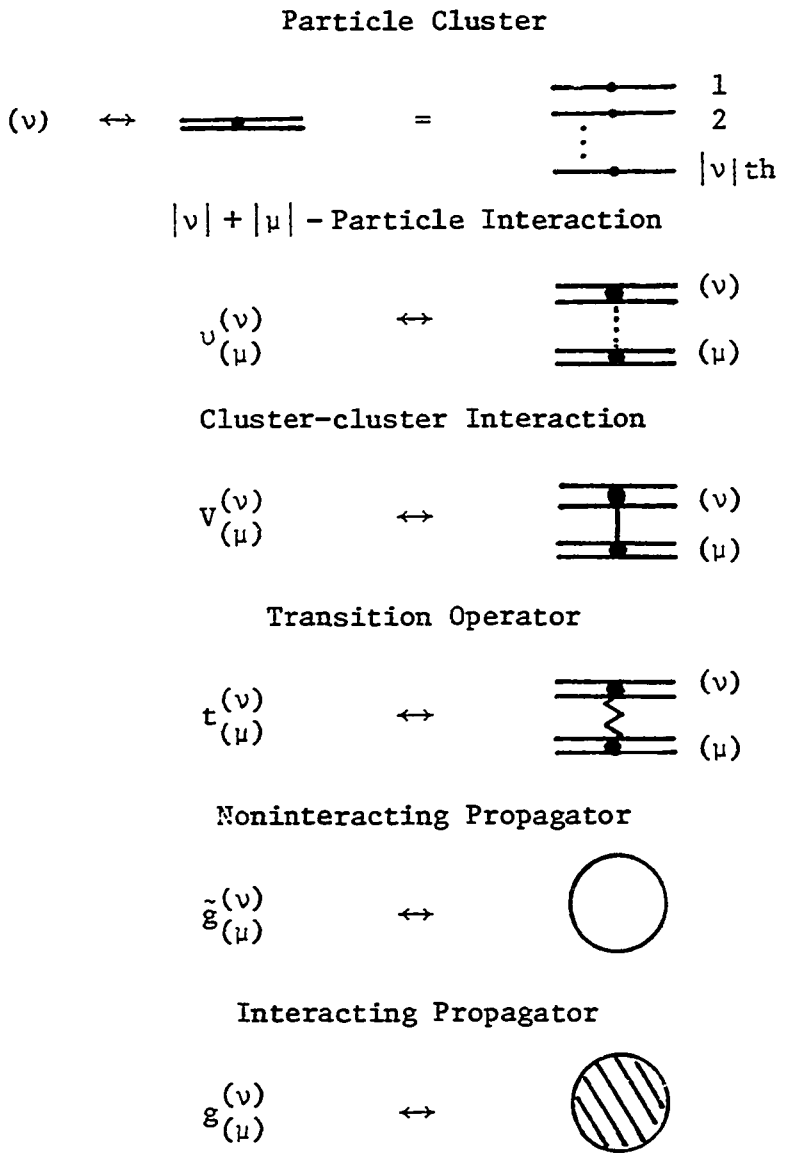
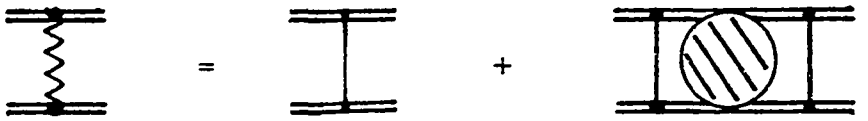
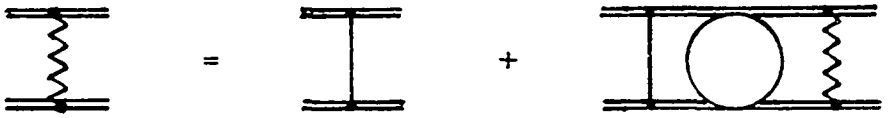


Figure 1. Operators and diagrammatic equivalents

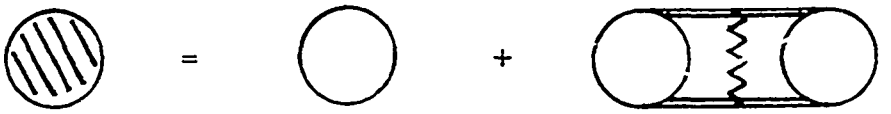
$t = V + VgV$



$t = V + V\tilde{g}t$



$g = \tilde{g} + \tilde{g}t\tilde{g}$



$g = \tilde{g} + \tilde{g}Vg$

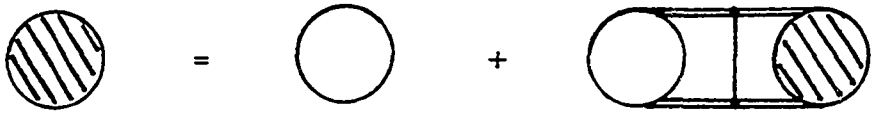


Figure 2. Operator expressions and diagrammatic equivalents

$$T(4) = \sum \tau_{\alpha}^{ijk} + \sum \tau_{\alpha\beta}^{ij} + \sum \tau_{\alpha\beta\gamma}^{i} \quad , \tag{2.54c}$$

$$\vdots$$

and

$$T(A + B) = \tau_{(B)}^{(A)} \quad , \tag{2.54d}$$

where

$$T = \sum_{n=2}^{(A+B)} T(n) \quad . \tag{2.55}$$

The physical content of the T(n)'s is clear. T(n) is the contribution in which n-particles participate in the interaction. Thus T(2) is the superposition of all pairwise scatterings of one particle from A with one particle from B. T(3) contains a pair from A(B) encountering one particle from B(A). Similarly, for T(4) we have the sum of contributions in which one particle from A(B) interacts with a triplet cluster from B(A), and those in which a correlated pair from A interacts with a correlated pair from B. The extension to arbitrary T(n) is apparent. For particular choices of $\tilde{g}_{(\mu)}^{(\nu)}$, the transition operators $\tau_{(\mu)}^{(\nu)}$ can be shown to be completely connected (1,10,26) with respect to the labeled particles and completely disconnected from the spectator particles. This fact allows us to view the T(n)'s as a perturbative sequence, in which we hope to obtain a good approximation to T by the use of the term T(2) with T(3), T(4), etc. providing the successive corrections.

The concept of 'connectivity' (11,12) is one which often appears in discussions of many-body scattering theory (26). An N-body operator θ_N is 'M-body connected' (denoted θ_{NM}) by particle interactions, provided it does not commute with any sum of the individual particle momentum operators $\vec{P}_i$ of the form

$$\vec{P}_M = \sum_i^M C_i \vec{P}_i \quad ; \quad C_i = 0, 1 \quad ,$$

such that,

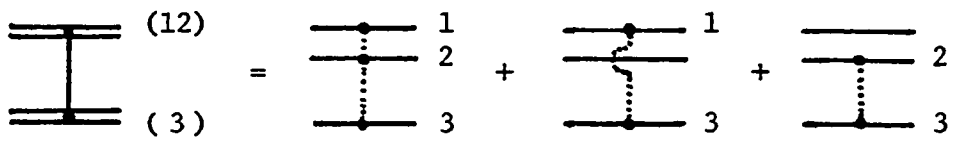
$$[\theta_{NM}, \vec{P}_M] \neq 0 \quad ,$$

where M is some distinct labeled subset of the N particles. For M = N an N-body operator is fully connected, but only partially connected for M < N. Figure 3 describes diagrammatically t_3^{12}, a disconnected 3-body operator. It is evident that iteration will produce an infinite set of disconnected particle lines. While t_3^{12} has a disconnected structure, the transition operator τ_3^{12} for $g_3^1 = g_3^2 = g_3^{12} = g$ is obviously fully connected and takes the form described in Fig. 4.

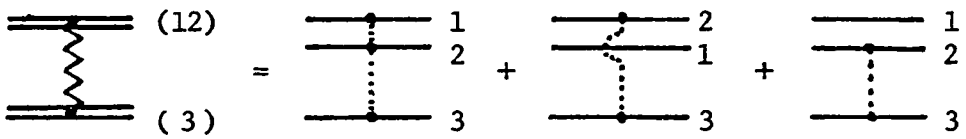
The scope of the identity expansion Eq. (2.50a) of T is evident through its reduction to the generalization of the Watson multiple scattering series for cluster scattering. Let all $\tilde{g}_{(\mu)}^{(\nu)} \equiv G_o$ in Eqs. (2.52a-e) and allow only two-body interactions. As the results in Appendix A show, the arbitrariness in $\tilde{g}_{(\mu)}^{(\nu)}$ permits this choice. Then we obtain

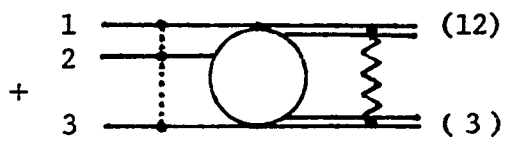
$$\bar{t}_\alpha^i = v_\alpha^i + v_\alpha^i G_o \bar{t}_\alpha^1 \quad . \tag{2.56}$$

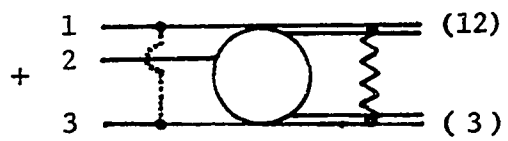
$$V_3^{12} = v_3^{12} + v_3^1 + v_3^2$$



$$t_3^{12} = V_3^{12} + V_3^{12}\, g_3^{12}\, t_3^{12}$$







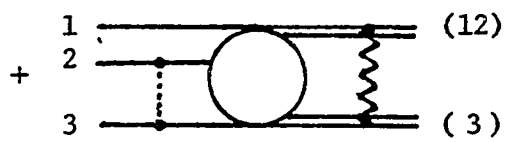


Figure 3. Example of disconnected operators

$$\tau_3^{12} = t_3^{12} - t_3^1 - t_3^2$$

$$= v_3^{12} + v_3^{12} \tilde{g} \tau^{12} + v_3^1 \tilde{g} t_3^{12}$$

$$+ v_3^2 \tilde{g} t_3^{12} + v_3^{12} \tilde{g} t_3^1 + v_3^2 \tilde{g} t_3^1$$

$$+ v_3^{12} \tilde{g} t_3^2 + v_3^1 \tilde{g} t_3^2$$

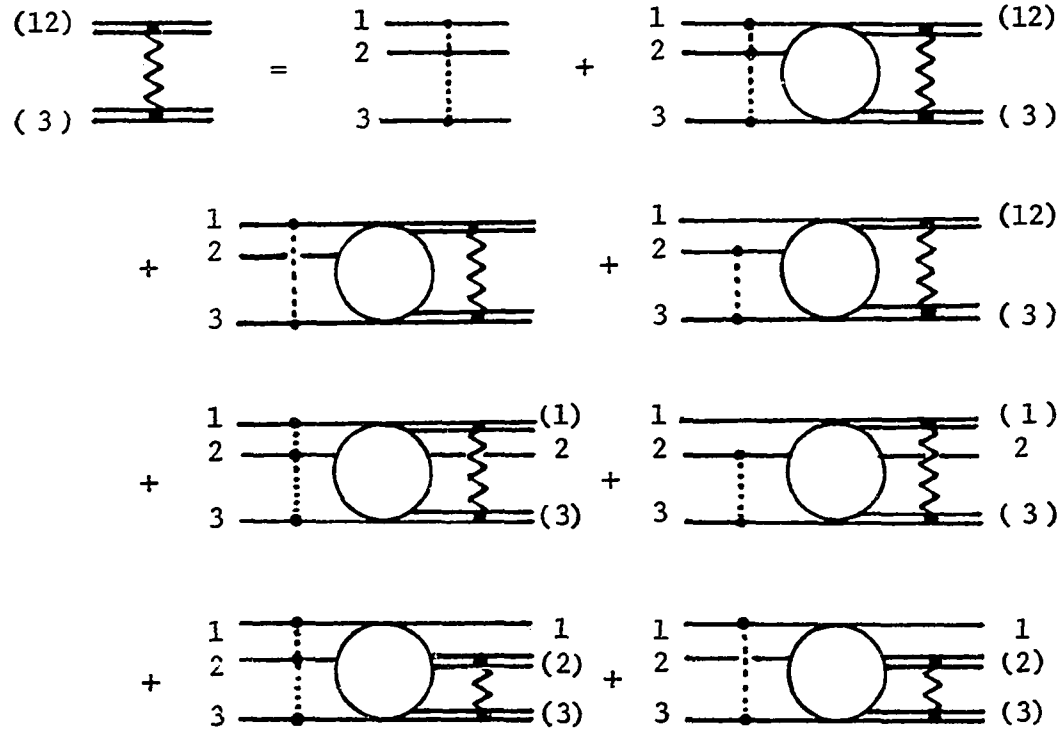


Figure 4. Example of fully connected operator

A careful examination of the expressions for $\tau_{(\mu)}^{(\nu)}$ in Eq. (2.50b) yields the following regrouping,

$$T = \sum \bar{t}_\alpha^1 + \sum \{\bar{t}_\alpha^1 G_o \bar{t}_\alpha^j + \bar{t}_\alpha^j G_o \bar{t}_\alpha^i + \bar{t}_\alpha^i G_o \bar{t}_\alpha^j G_o \bar{t}_\alpha^i + \bar{t}_\alpha^j G_o \bar{t}_\alpha^1 G_o \bar{t}_\alpha^j \cdots\}$$
$$+ \sum \{\bar{t}_\alpha^i G_o \bar{t}_\beta^i + \bar{t}_\beta^i G_o \bar{t}_\alpha^i + \bar{t}_\alpha^i G_o \bar{t}_\beta^i G_o \bar{t}_\alpha^i + \bar{t}_\beta^i G_o \bar{t}_\alpha^i G_o \bar{t}_\beta^i + \cdots\}$$
$$+ \sum \{\bar{t}_\alpha^i G_o \bar{t}_\alpha^j G_o \bar{t}_\alpha^k + \cdots\}$$
$$+ \cdots \quad . \tag{2.57}$$

Equation (2.57) may be reorganized to give

$$T = \sum \bar{t}_\alpha^i + \sum_{i\neq j} \bar{t}_\alpha^i G_o \bar{t}_\alpha^j + \sum_{\alpha\neq\beta} \bar{t}_\alpha^i G_o \bar{t}_\beta^i$$
$$+ \sum_{i\neq j,\ \alpha\neq\beta} \bar{t}_\alpha^i G_o \bar{t}_\beta^j + \sum_{i\neq j,\ j\neq k} \bar{t}_\alpha^i G_o \bar{t}_\alpha^j G_o \bar{t}_\alpha^k$$
$$+ \sum_{\alpha\neq\beta,\beta\neq\gamma} \bar{t}_\alpha^i G_o \bar{t}_\beta^i G_o \bar{t}_\gamma^i + \sum_{\substack{i\neq j, j\neq k \\ \alpha\neq\beta,\beta\neq\gamma}} \bar{t}_\alpha^i G_o \bar{g}_\beta^j G_o \bar{t}_\gamma^k + \cdots \quad . \tag{2.58}$$

Here Eq. (2.58) contains an infinite set of terms and for B = 1 reduces to

$$T(B=1) = \sum \bar{t}_1^i + \sum_{i\neq j} \bar{t}_1^i G_o \bar{t}_1^i + \sum_{i\neq j, j\neq k} \bar{t}_1^i G_o \bar{t}_1^j G_o \bar{t}^k + \cdots \quad , \tag{2.59}$$

which is the Watson multiple scattering series (14). Thus beginning with the expansion Eq. (2.50a) for T we have obtained in a straightforward fashion the generalization Eq. (2.58) of the Watson multiple-

scattering series Eq. (2.59). We observe that the content of T in terms of the identity expansion readily yields the connection with conventional multiple-scattering series. The flexibility inherent in our ability to choose the $\tilde{g}^{(\nu)}_{(\mu)}$ permits us to construct different expansions on the basis of utility and convenience. In brief, we have a method for examining cluster-cluster interactions in a systematic fashion, which readily displays the connection to the exact result stemming from different leading order approximations.

D. Selection of Subcluster Propagators

In Section C we decomposed the cluster-cluster T operator into a finite series of participant-spectator operators. While this expansion is not a 'perturbation' series in the standard sense of order-by-order in some coupling parameter, we hope that the selection of the subcluster propagators can be made so as to maximize the content of T(2) which is readily calculable and minimize the corrections due to T(3), T(4), etc. The correlation decomposition resembles the hole-line expansion (38,39) of nuclear-structure calculations which is effectively an expansion in the density times a correlation volume.

This section is concerned with developing a systematic set of definitions for the subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$ and $g^{(\nu)}_{(\mu)}$. We desire a set of $\tilde{g}^{(\nu)}_{(\mu)}$ for which the T(n) will satisfy one of the conditions of Eqs. (2.60a-d)

$$\langle T \rangle \cong \langle T(2) \rangle >> \langle \sum_{n=3}^{N} T(n) \rangle \tag{2.60a}$$

$$\langle T \rangle \cong \langle \sum_{n=2}^{m} T(n) \rangle >> \langle \sum_{k=m+1}^{N} T(k) \rangle \qquad (2.60b)$$

$$\langle T(2) \rangle >> \langle T(3) \rangle > \langle T(4) \rangle > \langle T(5) \rangle > \cdots > \langle T(N) \rangle \quad . \qquad (2.60c)$$

In particular applications it is possible that a specific subcluster-subcluster interaction provides the dominant character of the series, such as in collective scattering processes. Thus another convenient condition might be

$$\langle T \rangle \cong \langle T(m) \rangle >> \langle \sum_{n \neq m}^{N} T(n) \rangle \quad . \qquad (2.60d)$$

An example of such an application might be in treating nucleon-nucleon scattering as a collection of quarks scattering from a collection of quarks. Thus for nucleons described as bound states of three quarks, the scattering operator T may be dominated naturally by T(6). Another example might be the collisions of even-even N = Z nuclei. It may well be that the total amplitude is dominated by the alpha-alpha terms which are contained in T(8).

There is no a priori guarantee that a set of subcluster propagators exists for which any of the conditions of Eqs. (2.60a-d) hold. We shall assume, for practical purposes, that we can find a set for which Eq. (2.60a) or Eq. (2.60b) with m small is satisfied. Much work has been done in which one or more of these conditions appears to have been fulfilled with satisfactory results. To cite just one example consider the work of Fujita and Hüfner (40) in which they develop a phenomenological treatment of back-angle proton scattering from nuclei.

They find that the differential cross section appears to be the superposition of scatterings from single target nucleons, correlated pairs, correlated triplets, and so forth. Such a description demonstrates an application where condition Eq. (2.60b) appears to be satisfied.

In what follows we hope to motivate choices of the propagators which reduce the computational magnitude of the 'lower order' T(n) by insuring that the respective $t^{(\nu)}_{(\mu)}$'s are effective n(= $|\nu| + |\mu|$)-body operators rather than N-body operators. Furthermore, we believe such choices should possess transparent physical interpretations. For example in T(2), we would like a choice of the subcluster propagators $\tilde{g}^i_\alpha$ which reduces the t^i_α to effective two-body operators rather than being a fully coupled N-body operator. Similarly in T(3), we desire fully-coupled three-body operators, 'disconnected' from the remaining N - 3 objects. Or generally, T(n) is to be an n-body operator which is diagonal in the space of the remaining N - n particles.

By using Lemma A we can expand H_o, the noninteracting cluster-cluster Hamiltonian, in terms of arbitrary partial Hamiltonians $h^{(\nu)}_{(\mu)}$ as

$$H_o = \sum h^i_\alpha + \sum (h^{ij}_\alpha - h^i_\alpha - h^j_\alpha) + \sum (h^i_{\alpha\beta} - h^i_\alpha - h^i_\beta) + \cdots \quad . \tag{2.61a}$$

$$= \sum H^i_\alpha + \sum H^{ij}_\alpha + \sum H^i_{\alpha\beta} + \cdots \quad , \tag{2.61b}$$

with the restriction that

$$h^{(A)}_{(B)} \equiv H_o \quad . \tag{2.62}$$

It is then natural to use the $h^{(\nu)}_{(\mu)}$ selected for a given application to

define the hitherto arbitrary $\tilde{g}^{(\nu)}_{(\mu)}$ by

$$\tilde{g}^{(\nu)}_{(\mu)} \equiv (Z - h^{(\nu)}_{(\mu)})^{-1} \quad . \tag{2.63a}$$

Here Z represents the complex parametric energy which includes the appropriate boundary conditions. From Eq. (2.63) and the resolvent Eqs. (2.53a,b) we obtain for the interacting subcluster-subcluster propagators $g^{(\nu)}_{(\mu)}$, the expression

$$g^{(\nu)}_{(\mu)} \equiv (Z - h^{(\nu)}_{(\mu)} - V^{(\nu)}_{(\mu)})^{-1} \quad . \tag{2.63b}$$

Note that, in terms of the partial Hamiltonians $h^{(\nu)}_{(\mu)}$, $g^{(A)}_{(B)} = (Z - h^{(A)}_{(B)} - V^{(A)}_{(B)})^{-1} = G$, and that $\tilde{g}^{(A)}_{(B)} = (Z - h^{(A)}_{(B)})^{-1} = G_o$, which are the proper limits of $g^{(\nu)}_{(\mu)}$ and $\tilde{g}^{(\nu)}_{(\mu)}$ respectively.

In order for the $t^{(\nu)}_{(\mu)}$ to operate upon the participants (i.e., the labeled particles), the $h^{(\nu)}_{(\mu)}$ must be defined to contain the $(\nu) + (\mu)$ particles; the remaining particles may appear provided no interactions are permitted which link the two groups. We make the following definitions,

$$\bar{h}^{(\nu)} \equiv \sum_{i \subset (\nu)} h^i + V^{(\nu)} \quad , \tag{2.64a}$$

$$\bar{h}_{(\mu)} \equiv \sum_{\alpha \subset (\mu)} h_\alpha + V_{(\mu)} \quad , \tag{2.64b}$$

$$\bar{h}^{(\nu)}_{(\mu)} = \bar{h}^{(\nu)} + \bar{h}_{(\mu)} \quad . \tag{2.65}$$

The $\bar{h}^{(\nu)}$ and $\bar{h}_{(\mu)}$ are the subcluster Hamiltonians of the (ν) and (μ)

subclusters of A and B respectively. We can write the residual subcluster Hamiltonians as $\bar{h}^{(A-\nu)}$ and $\bar{h}_{(B-\mu)}$. Equation (2.65) defines the noninteracting subcluster-subcluster Hamiltonian, $\bar{\bar{h}}^{(\nu)}_{(\mu)}$.

We are free to define the partial Hamiltonian $h^{(\nu)}_{(\mu)}$ by the following expression

$$h^{(\nu)}_{(\mu)} \equiv \bar{\bar{h}}^{(\nu)}_{(\mu)} + \bar{\bar{h}}^{(A-\nu)}_{(B-\mu)} \quad , \tag{2.66}$$

which obeys the restriction of Lemma A, that $h^{(A)}_{(B)} \equiv H_o$, as can be seen by examining the definitions Eqs. (2.64a,b) and (2.65). It is important to emphasize that the definition Eq. (2.66) specifically leaves out interactions between elements of (ν) and $(A-\nu)$ as well as those between (μ) and $(B-\mu)$. In this way the definitions insure the spectator separation of the expansions. It is possible to introduce into these definitions auxiliary potentials. These may arise from mean-field definitions or may merely be physically or computationally desirable. The only restriction is that such fields must become zero for $(\nu) = (A)$ and $(\mu) = (B)$.

An alternative definition for the partial Hamiltonian $h^{(\nu)}_{(\mu)}$, which also maintains the separate character of the participants and spectators, is apparent from the definition of the subcluster-subcluster Hamiltonian $\bar{\bar{h}}^{(\nu)}_{(\mu)}$. We may choose

$$h^{(\nu)}_{(\mu)} \equiv \bar{\bar{h}}^{(\nu)}_{(\mu)} + \Delta\varepsilon^{(\nu)}_{(\mu)} \quad , \tag{2.67}$$

where

$$\Delta\varepsilon^{(A)}_{(B)} \equiv 0 \quad . \tag{2.68}$$

The $\Delta\varepsilon^{(\nu)}_{(\mu)}$ may be chosen to represent the energy shift due to the residual particles. It is readily verified that Eq. (2.67) obeys the restriction of Eq. (2.64).

The definitions Eq. (2.66) of the partial Hamiltonians in conjunction with the definitions Eq. (2.63a) of the propagators $\tilde{g}^{(\nu)}_{(\mu)}$ and Eq. (2.52a) for the partial transition operators $t^{(\nu)}_{(\mu)}$, have the virtue that the $t^{(\nu)}_{(\mu)}$ become effective $n(= |\nu| + |\mu|)$-body operators which are diagonal with respect to the residual Hamiltonian particles (spectators). Similarly, the definition Eq. (2.67) yields $t^{(\nu)}_{(\mu)}$ which are true n-body operators. Other definitions for the partial Hamiltonians $h^{(\nu)}_{(\mu)}$ are possible.

It is of some interest to note that the partial transition operators $t^{(\nu)}_{(\mu)}$ corresponding to the partial Hamiltonians $h^{(\nu)}_{(\mu)}$ of Eq. (2.67) with the choice $\Delta\varepsilon^{(\nu)}_{(\mu)} \equiv 0$ and in the absence of mean fields (see Section B) are the free (ν)-cluster (μ)-cluster transition operators. For example, t^{i}_{α} is then the free nucleon-nucleon transition operator, if i and α are nucleons. Thus for $\Delta\varepsilon^{i}_{\alpha} \neq 0$, t^{i}_{α} is an energy-shifted two-body t operator; we may say that $\Delta\varepsilon^{i}_{\alpha}$ accounts in some average fashion for the off-shell nature of t^{i}_{α} when i and α are embedded in larger groups of particles. Including mean fields in $h^{(\nu)}_{(\mu)}$ permits a more complete treatment of the off-shell dynamics and renders t^{i} an off-shell two-nucleon t-matrix.

The set of definitions Eq. (2.67) for the partial Hamiltonians yields the generalization of the propagators defined by ST (24); this selection also corresponds in lowest order to the closure approximation

used by Ernst, Londergan, Miller and Thaler, (ELMT) (23). In the work of ELMT, the T-operator is approximated by replacing all of the $\tilde{g}^{(\nu)}_{(\mu)}$'s by the choice for $\tilde{g}^{i}_{\alpha}$ and retaining only two-body interactions. To obtain some systematic order-by-order estimate of the discrepancy between T and T_{ELMT}, we would look at $\langle T(n) - T(n)_{ELMT} \rangle$. For n = 2, $T(2) = T(2)_{ELMT}$.

We may give a spectral representation for the partial propagator $\tilde{g}^{(\nu)}_{(\mu)}$, if we first construct the scattering solutions to the subcluster Hamiltonians.

$$[\varepsilon^{(\nu)}_{(\mu)}(a) - \bar{h}^{(\nu)}_{(\mu)}]\ \phi^{(\nu)}_{(\mu)}(a) = 0 \tag{2.69a}$$

$$[\varepsilon^{(\nu)}_{(\mu)}(a) - \bar{h}^{(\nu)}_{(\mu)} - V^{(\nu)}_{(\mu)}]\ \Psi^{(\pm)(\nu)}_{(\mu)}(a) = 0 \tag{2.69b}$$

$$[E^{(\nu)}(b) - \bar{h}^{(A-\nu)}]\ \chi^{(A-\nu)}(b) = 0 \tag{2.70a}$$

$$[E_{(\mu)}(c) - \bar{h}_{(B-\mu)}]\ \chi_{(B-\mu)}(c) = 0 \tag{2.70b}$$

Thus for the definition Eq. (2.67) of the partial Hamiltonian, $h^{(\nu)}_{(\mu)}$ the spectral decompositions of the subcluster-subcluster propagators $\tilde{g}^{(\nu)}_{(\mu)}$ and $g^{(\nu)}_{(\mu)}$ are

$$\tilde{g}^{(\nu)}_{(\mu)} = \sum_{a} \frac{|\phi^{(\nu)}_{(\mu)}(a)\rangle\langle\phi^{(\nu)}_{(\mu)}(a)|}{Z - \varepsilon^{(\nu)}_{(\mu)}(a) - \Delta\varepsilon^{(\nu)}_{(\mu)}}\ , \tag{2.71}$$

and

$$g^{(\nu)}_{(\mu)} = \sum_a \frac{|\Psi^{(\pm)(\nu)}_{(\mu)}(a)\rangle\langle\Psi^{(\pm)(\nu)}_{(\mu)}(a)|}{Z - \varepsilon^{(\nu)}_{(\mu)}(a) - \Delta\varepsilon^{(\nu)}_{(\mu)}} \quad . \tag{2.72}$$

We may also use definition Eq. (2.66), in which case we obtain the expression

$$\tilde{g}^{(\nu)'}_{(\mu)} = \sum_{a,b,c} \frac{S|\phi^{(\nu)}_{(\mu)}(a)\chi^{(A-\nu)}(b)\chi_{(B-\mu)}(c)\rangle\langle\phi^{(\nu)}_{(\mu)}(a)\chi^{(A-\nu)}(b)\chi_{(B-\mu)}(c)|}{Z - \varepsilon^{(\nu)}_{(\mu)}(a) - E^{(\nu)}(b) - E_{(\mu)}(c)} \quad , \tag{2.73}$$

with an analogous expression for $g^{(\nu)'}_{(\mu)}$. Here S is the appropriate symmetrization operator. For example, if all N-particles are nucleons, then S antisymmetrizes the particles between the state vectors $\phi^{(\nu)}_{(\mu)}$, $\chi^{(A-\nu)}$ and $\chi_{(B-\mu)}$, unless the proper symmetrization has already been included in the intercluster potentials $v^{(\nu)}_{(\mu)}$ (28).

E. K-Matrix and Optical Potential Applications

In reducing the N-body problem to systems of fewer than N-bodies, we have not addressed all the physical and practical aspects that arise. For example, in applying the GSE to nuclear systems we have not considered specifically the effect of the Pauli principle. We now focus our attention on the nuclear applications and address the questions of the Pauli principle, the strong repulsive core of the nucleon-nucleon interaction (V_{NN}) and the reduction to calculable spaces. For bound states of finite nuclei the effective two-nucleon interaction has been derived from a free potential V_{NN} through the Brueckner G-matrix,

$$G_B(\omega) = V_{NN} + V_{NN} \frac{Q_p}{\omega - h_o(1) - h_o(2)} G_B(\omega) \quad ,$$

where h_o is a single particle Hamiltonian incorporating the kinetic energy and a single particle potential for each of the interacting nucleons. The G-matrix is a renormalized nucleon-nucleon interaction which, through the infinite sum of potential scatterings, eliminates the strong effects of the repulsive core in much the same manner as does the t-matrix. The Pauli operator Q_p prevents the interacting nucleon pair from scattering into states which are already occupied by the remaining nucleons or which are treated explicitly in the dynamical framework (e.g., diagonalization). For all applications Q_p must be specified in the selected basis representation. The two most common bases for nuclear problems are the harmonic oscillator and the plane wave representation.

The development in this section uses K rather than G_B or G to avoid notational confusion. I develop a formal relationship between the T and K operator expansions of GSE. From Appendix B we know that given projectors (41), P + Q = 1, the operator expression (2.74)

$$A = B + BCD \quad , \tag{2.74}$$

may be rewritten as

$$A = D + DCPA \quad , \tag{2.75}$$

where

$$D = B + BDQD \quad . \tag{2.76}$$

The general expressions Eqs. (2.75) and (2.76) together possess the same content as Eq. (2.74) and may be used to generate specific operator separations. The subcluster transition operators $t^{(\nu)}_{(\mu)}$ of Eq. (2.52) are defined by operator expressions of the form of Eq. (2.72). Thus if I define a set of projectors $\{p^{(\nu)}_{(\mu)}, q^{(\nu)}_{(\mu)}\}$ to obey the projection rules of Appendix B, I may define a new set of operators $\{k^{(\nu)}_{(\mu)}\}$ by

$$k^{(\nu)}_{(\mu)} = \bar{v}^{(\nu)}_{(\mu)} + v^{(\nu)}_{(\mu)} \tilde{g}^{(\nu)}_{(\mu)} q^{(\nu)}_{(\mu)} k^{(\nu)}_{(\mu)} \quad , \tag{2.77}$$

which are related to the transition operators $t^{(\nu)}_{(\mu)}$ by the expression

$$t^{(\nu)}_{(\mu)} = k^{(\nu)}_{(\mu)} + k^{(\nu)}_{(\mu)} \tilde{g}^{(\nu)}_{(\mu)} p^{(\nu)}_{(\mu)} t^{(\nu)}_{(\mu)} \quad . \tag{2.78}$$

with the restriction that

$$p^{(\nu)}_{(\mu)} + q^{(\nu)}_{(\mu)} = 1(n) \quad . \tag{2.79}$$

The Eqs. (2.77) and (2.78) are formally equivalent to the expressions Eqs. (2.51) for the subcluster transition operator.

I define a total reaction operator K by

$$K = \sum k^{i}_{\alpha} + \sum (k^{ij}_{\alpha} - k^{i}_{\alpha} - k^{j}_{\alpha}) + \cdots \quad . \tag{2.80}$$

The physical content of Eq. (2.80) is manifest once the subcluster projectors $q^{(\nu)}_{(\mu)}$ have been specified. Two examples we consider in this section involve selections of the $q^{(\nu)}_{(\mu)}$ such that K is an optical potential, or such that K = T. This latter condition implies only that $q^{(A)}_{(B)} \equiv 1(N)$. The remaining $q^{(\nu)}_{(\mu)}$ are still at our disposal to specify. Using the definitions of the Brueckner reaction matrix as a guide, the

$q^{(\nu)}_{(\mu)}$ are selected to have the property that states occupied by the $|A+B-\nu-\mu|$-spectators are forbidden to the $|\nu+\mu|$-participants. In this case $k^{(\nu)}_{(\mu)}$ may be called the subcluster reaction operator.

In order to illustrate the physical content of the $k^{(\nu)}_{(\mu)}$, I specify a set of $q^{(\nu)}_{(\mu)}$ in the spectral representation of Eqs. (2.71) or (2.73). The Pauli projector is defined as

$$q^{(\nu)}_{(\mu)} = \sum |\phi^{(\nu)}_{(\mu)}(a)\rangle\langle\phi^{(\nu)}_{(\mu)}(a)| \quad . \tag{2.81}$$

(a) Pauli-restricted by $|A+B-\nu-\mu|$-spectators

Note that for $(\nu) = (A)$ and $(\mu) = (B)$, $q^{(A)}_{(B)} = 1(N)$, since the absence of spectators eliminates the additional external Pauli restrictions upon (a). Of course, whatever contributions are not included in $q^{(\nu)}_{(\mu)}$ are implicitly contained in $p^{(\nu)}_{(\mu)}$. From a pragmatic viewpoint the content of $q^{(\nu)}_{(\mu)}$ may be enlarged or restricted in order to construct the model space in which $p^{(\nu)}_{(\mu)}$ is desired to operate. However, we must remark, that while such modifications do not affect the validity of the expansion for K, the convergence of the lower-order terms may be affected.

As mentioned above we can also construct an optical potential through the judicious selection of P and Q. In order to maintain a close parallel to the work of ELMT and ST, we shall develop a generalization of the optical potential discussed by ST. We select $P + Q = 1(N)$, where P is the projection onto the nuclear ground states of clusters A and B. Thus for $\tilde{g}^{i}_{\alpha}$, $\tilde{g}^{ij}_{\alpha}$, $\cdots$, P and Q operate in a sequence of different basis representations but retain the same physical content. In order to emphasize this point I use P_{opt} and Q_{opt} throughout rather than $P^{(\nu)}_{opt(\mu)}$

and $Q_{opt(\mu)}^{(\nu)}$. From Eq. (2.52) we have

$$t_{(\mu)}^{(\nu)} = V_{(\mu)}^{(\nu)} + V_{(\mu)}^{(\nu)} \tilde{g}_{(\mu)}^{(\nu)} t_{(\mu)}^{(\nu)} \quad ,$$

and in analogy with Eqs. (2.77) and (2.78) we obtain

$$u_{(\mu)}^{(\nu)} = V_{(\mu)}^{(\nu)} + V_{(\mu)}^{(\nu)} \tilde{g}_{(\mu)}^{(\nu)} Q_{opt} u_{(\mu)}^{(\nu)} \tag{2.82}$$

and

$$t_{(\mu)}^{(\nu)} = u_{(\mu)}^{(\nu)} + u_{(\mu)}^{(\nu)} \tilde{g}_{(\mu)}^{(\nu)} P_{opt} t_{(\mu)}^{(\nu)} \quad . \tag{2.83}$$

I may now write the total optical potential as

$$U_{opt} = \sum u_\alpha^i + \sum (u_\alpha^{ij} - u_\alpha^i - u_\alpha^j) + \sum (u_{\alpha\beta}^i - u_\alpha^i - u_\alpha^i) + \cdots \tag{2.84a}$$

$$= \sum U_\alpha^i + U_\alpha^{ij} + \sum U_{\alpha\beta}^i + \cdots \quad . \tag{2.84b}$$

Recall from Lemma A that $U_{opt} \equiv u_{(B)}^{(A)}$, and from Eq. (2.82) we obtain

$$u_{(B)}^{(A)} = V_{(B)}^{(A)} + V_{(B)}^{(A)} G_o Q_{opt} u_{(B)}^{(A)} \quad ,$$

which is the conventional defintion of U_{opt}. In correspondence to Eq. (2.54) for T(n), we now write

$$U_{opt}(2) \equiv \sum U_\alpha^i \quad , \tag{2.85a}$$

$$U_{opt}(3) \equiv \sum U_\alpha^{ij} + \sum U_{\alpha\beta}^i \quad , \tag{2.85b}$$

$$\vdots$$

$$U_{opt}(n) \equiv \sum_{\ell=1}^{n-1} \sum_{\substack{(\nu)\subset A \\ (\mu)\subset B}} \sum_{\substack{|\nu|=\ell \\ |\mu|=n-\ell}} U_{(\mu)}^{(\nu)} \quad . \tag{2.85c}$$

Equations (2.84a-c) let us write the total optical potential as

$$U_{opt} = \sum_{n=2}^{A+B} U_{opt}(n) \quad . \tag{2.86}$$

It is easy to recognize the physical content of the $U_{opt}(n)$ for this expansion. $U_{opt}(n)$ is the contribution to the optical potential of n-particles scattering in all possible combinatorial patterns. Specifically, U(2) is the superposition of all pairwise scattering between a particle from A and a particle from B, in which both cluster A and B remain in their ground state configuration. For U(3), we have the sum of the correlated pairs from A(B) scattering from a single particle from B(A) and again both clusters A and B remain in their respective ground states. And so forth for U(n).

We recognize that the exact content of $U_{opt}(n)$ depends upon the subcluster propagators $\tilde{g}_{(\mu)}^{(\nu)}$. Thus, for different propagators, the exact contribution of $U_{opt}(n)$ to U_{opt} will change. Furthermore while we have specifically retained the conventional definition of U_{opt} in regard to defining Q_{opt} and P_{opt}, we could in fact obtain a different set of $U'_{opt}(n)$ which give U_{opt} for an alternate view of Q and P. For example, if we define the set of $\{p_{(\mu)}^{(\nu)}\}$ as the projectors upon the ground states of the subclusters $(A-\nu)$ and $(B-\mu)$, then $U'_{opt} \equiv U_{opt}$ but in general $U'_{opt}(n) \neq U_{opt}(n)$.

F. Matrix Elements

The bulk of my discussion has been concerned with the development of formal operator expressions and with the selection of subcluster operators based upon physical arguments. In this section I am more directly concerned with the construction of matrix elements for these operators. I consider specifically the elastic scattering of two nuclei wherein only two-body potentials are retained.

As has been emphasized by ST, it is only in the case where t_α^i is a two-body operator that the T(2) matrix elements reduce to a simple form. In the following discussion the indices i and α are suppressed where no confusion can arise. Furthermore, I restrict this matrix element development to the case where g_α^i is the sum of simple kinetic energy operators for the interacting nucleons as in Eq. (2.71). Thus I can write

$$\tilde{g}(\vec{p},\vec{k}) = (z - \vec{p}^2/2m - \vec{k}^2/2m)^{-1} \quad . \tag{2.87}$$

Here $\vec{p}(\vec{k})$ refers to the momentum vector of the nucleon from A(B). We may now write for the two-body operator t of Eq. (2.52) the expression

$$(\vec{p}'\vec{k}'|t|\vec{p}\vec{k}) = (\vec{p}'\vec{k}'|v|\vec{p}\vec{k}) + \int d\vec{p}''d\vec{k}''(\vec{p}'\vec{k}'|v|\vec{p}''\vec{k}'')\tilde{g}(\vec{p}'',\vec{k}'')(\vec{p}''\vec{k}''|t|\vec{p}\vec{k}) \quad . \tag{2.88}$$

Let $\vec{q}$ and $\vec{P}_c$ be defined as follows,

$$\vec{q} = (\vec{p} - \vec{k})/2 \tag{2.89a}$$

$$\vec{P}_c = \vec{p} + \vec{k} \tag{2.89b}$$

then t, for translationally invariant interaction v, may be written as

$$(\vec{q}'|t|\vec{q}) = (\vec{q}'|v|\vec{q}) + \int d\vec{p}''(\vec{q}'|v|\vec{q}'') \frac{1}{z - \vec{P}_c^2/4m - \vec{q}''^2/m} (\vec{q}''|t|\vec{q}) \quad . \quad (2.90)$$

In order to connect the two-nucleon matrix elements to the scattering system, I define the relationship between the interacting pair and the remaining (passive) nucleons. Let $\vec{P}_{cm}$ and $\vec{P}_r$ be the total momentum and relative momentum of the A - B system. $\vec{P}_A(\vec{P}_B)$ is defined as the total momentum of the cluster A(B), and $\vec{P}_{Ai}(\vec{P}_{B\alpha})$ as the total momentum of the cluster A(B) with the i-th (α-th) nucleon removed. We then have the following definitions.

$$\vec{P}_A = \sum_j^A \vec{p}^j = \vec{p}^i + \sum_{j\neq i}^A \vec{p}^j = \vec{p}^i + \vec{P}_{Ai} \quad (2.91a)$$

$$\vec{P}_B = \sum_\alpha^B \vec{k}_\alpha = \vec{k} + \sum_{\alpha\neq\beta}^B \vec{k}_\beta = \vec{k}_\alpha + \vec{P}_{B\alpha} \quad (2.91b)$$

$$\gamma_A = M_A/(M_A + M_B) \quad (2.92a)$$

$$\gamma_B = M_B(M_A + M_B) \quad (2.92b)$$

$$\vec{P}_{cm} = \vec{P}_A + \vec{P}_B \quad (2.93)$$

$$\vec{P}_r = \gamma_B\vec{P}_A - \gamma_A\vec{P}_B \quad (2.94)$$

$$\vec{P}_A = \gamma_A\vec{P}_{cm} + \vec{P}_r \quad (2.95a)$$

$$\vec{P}_B = \gamma_B\vec{P}_{cm} - \vec{P}_r \quad (2.95b)$$

Here $M_A(M_B)$ is the mass of the nucleus A(B), and the $\vec{p}^j(\vec{k}_\beta)$ are the individual momentum vectors of the nucleons.

Consider an arbitrary two-body operator θ_α^i; in order to form the matrix elements of θ_α^i, it is first necessary to define certain quantities associated with the Hamiltonians of clusters A and B respectively. From Eqs. (2.70a) and (2.70b) I denote the total eigenvector of cluster A in eigenstate a by $\chi^A(a)$ and similarly for cluster B, $\chi_B(b)$. Associated with these eigenvectors are the intrinsic wavefunctions ($\phi_A^{IWF}(a)$ and $\phi_B^{IWF}(b)$) and the total momentum vectors ($\vec{P}_A$ and $\vec{P}_B$). Explicitly these are related by

$$|\chi^A(a)\rangle = |\vec{P}_A\ \phi_A^{IWF}(a)\rangle \tag{2.96a}$$

and

$$|\chi_B(b)\rangle = |\vec{P}_B\ \phi_B^{IWF}(b)\rangle \quad . \tag{2.96b}$$

Let the intrinsic momentum vectors be labeled $\vec{p}_i^{int}$ and $\vec{k}_j^{int}$ for A and B respectively. Then the momentum space realizations of the IWF are

$$\phi_A(a) = \langle \vec{p}_1^{int}\ \vec{p}_2^{int} \cdots \vec{p}_{A-1}^{int} | \phi_A^{IWF}(a)\rangle \tag{2.97a}$$

and

$$\phi_B(b) = \langle \vec{k}_1^{int}\ \vec{k}_2^{int} \cdots \vec{k}_{B-1}^{int} | \phi_B^{IWF}(b)\rangle \quad . \tag{2.97b}$$

Associated with the intrinsic wavefunctions is the possibility of separating one-particle from the cluster and viewing it independently of the remaining particles. The wavevectors associated with this

concept are called the reduced intrinsic wavefunctions RIWF and are denoted by $\Phi^{RIWF}_{Ai}(a)$ and $\Phi^{RIWF}_{B\alpha}(b)$, where particles i and α from A and B respectively are isolated for examination. The total momentum vectors of the spectator particles are then $\vec{P}_{Ai}$ and $\vec{P}_{B\alpha}$, and possess intrinsic momentum vectors labeled with primes. Thus

$$\phi_{Ai}(a) = \langle \vec{P}_{Ai}; \vec{p}^{int'}_1 \cdots \vec{p}^{int'}_{A-2} | \Phi^{RIWF}_{Ai}(a) \rangle \tag{2.98a}$$

$$\phi_{B\alpha}(b) = \langle \vec{P}_{B\alpha}; \vec{k}^{int'}_1 \cdots \vec{k}^{int'}_{B-2} | \Phi^{RIWF}_{B}(b) \rangle \quad . \tag{2.98b}$$

The transformations which connect the IWF and RIWF are defined by the expressions,

$$|\vec{P}_A \, \phi_A(a)\rangle = \int d\vec{p}^{\,i} \, \delta(\vec{p}^{\,i} + \vec{P}_{Ai} - \vec{P}_A) |\vec{p}^{\,i} \, \phi_{Ai}(a)\rangle \quad . \tag{2.99a}$$

The expression in Eq. (2.99a) is an integral transformation, since in general the $\phi_{Ai}(a)$ cannot be expected to be eigenstates of the spectator Hamiltonian. A similar transformation exists for cluster B and is written

$$|\vec{P}_B \, \phi_B(b)\rangle = \int d\vec{k}_\alpha \, \delta(\vec{k}_\alpha + \vec{P}_{B\alpha} - \vec{P}_B) |\vec{k}_\alpha \, \phi_{B\alpha}(b)\rangle \quad . \tag{2.99b}$$

The single body densities of the RIWF are defined as

$$\rho^{(1)}_{Ai}(\vec{P}_A - p^i)aa' = \int \phi^+_{Ai}(a) \, \delta(\vec{P}_{Ai} + \vec{p}^{\,i} - \vec{P}_A) \phi_{Ai}(a') d\vec{P}_{Ai} \, d\vec{p}^{\,int'}_1 \cdots d\vec{p}^{\,int'}_{A-2} \tag{2.100a}$$

$$\rho^{(1)}_{B\alpha}(\vec{P}_B - \vec{k}_\alpha)bb' = \int \phi^+_{B\alpha}(b) \, \delta(\vec{P}_{B\alpha} + \vec{k}_\alpha - \vec{P}_B) \phi_{B\alpha}(b') d\vec{P}_{Ai} \, d\vec{k}^{\,int'}_1 \cdots d\vec{k}^{\,int'}_{B-2} . \tag{2.100b}$$

The integration of these densities gives

$$\int \rho^{(1)}_{Ai}(\vec{p})_{aa'} d\vec{p} = {}_{aa'} \tag{2.101a}$$

$$\int \rho^{(1)}_{B\alpha}(\vec{P})_{bb'} d\vec{P} = \delta_{bb'} \quad . \tag{2.101b}$$

Having made these definitions and restrictions, the transformations for the A - B system can be written in the following manner.

$$\begin{aligned} |\vec{P}_{cm}\ \vec{P}_r\ \phi_A(0)\ \phi_B(0)\rangle = & \int d\vec{p}^{\,i}\ \delta(\vec{p}^{\,i} + \vec{P}_{Ai} - \gamma_A \vec{P}_{cm} - \vec{P}_r) \\ & \times d\vec{k}_\alpha\ \delta(\vec{k}_\alpha + \vec{P}_{B\alpha} - \gamma_B \vec{P}_{cm} + \vec{P}_r) \\ & \times |\vec{p}^{\,i}\ \vec{k}_\alpha\ \phi_{Ai}(0)\ \phi_{B\alpha}(0)\rangle \quad . \end{aligned} \tag{2.102}$$

Since we anticipate treating anti-symmetrization through effective multi-body interactions (28) we work in a simple product representation. Thus the matrix elements of the two-body operator θ^i_α can now be written as

$$\begin{aligned} &\langle \vec{P}^F_{cm}\ \vec{P}^F_r\ \phi_A(0)\ \phi_B(0)\ \theta^i_\alpha\ \vec{P}^I_{cm}\ \vec{P}^I_r\ \phi_A(0)\ \phi_B(0)\rangle = \\ &\int \langle \vec{p}^{\,iF}\ \vec{k}^F\ \phi_{Ai}(0)\ \phi_{B\alpha}(0)| d\vec{p}^{\,iF}\ \delta(\vec{p}^{\,iF} + \vec{P}_{Ai} - \gamma_A \vec{P}^F_{cm} - \vec{P}^F_r) \\ &\times d\vec{k}^F_\alpha\ \delta(\vec{k}^F_\alpha + \vec{P}_{B\alpha} - \gamma_B \vec{P}^F_{cm} + \vec{P}^F_r)\theta^i_\alpha\ d\vec{p}^{\,iI}\ \delta(\vec{p}^{\,iI} + \vec{P}_{Ai} - \gamma_A \vec{P}^I_{cm} - \vec{P}^I_r) \\ &\times d\vec{k}^I_\alpha\ \delta(\vec{k}^I_\alpha + \vec{P}_{B\alpha} - \gamma_B \vec{P}^I_{cm} + \vec{P}^I_r)|\vec{p}^{\,iI}\ \vec{k}^I_\alpha\ \phi_{Ai}(0)\ \phi_{B\alpha}(0)\rangle \quad . \end{aligned} \tag{2.103}$$

This expression reduces to

$$\langle \theta_\alpha^i \rangle = \int d\vec{p}^{\,iI}\, d\vec{k}_\alpha^{\,I} (\vec{p}^{\,iF}\, \vec{k}_\alpha^{\,F} | \theta_\alpha^i | \vec{p}^{\,iI}\, \vec{k}_\alpha^{\,I})$$

$$\times \rho_{Ai}^{(1)}(\vec{p}^{\,iI} - \gamma_A \vec{P}_{cm}^{\,I} - \vec{P}_r^{\,I}) \rho_{B\alpha}^{(1)}(\vec{k}_\alpha^{\,I} - \gamma_B \vec{P}_{cm}^{\,I} + \vec{P}_r^{\,I}0 \quad , \quad (2.104)$$

where

$$\vec{p}^{\,iF} = \vec{p}^{\,iI} + \gamma_A(\vec{P}_{cm}^{\,F} - \vec{P}_{cm}^{\,I}) + (\vec{P}_r^{\,F} - \vec{P}_r^{\,I}) \quad (2.105a)$$

$$\vec{k}_\alpha^{\,F} = \vec{k}_\alpha^{\,I} + \gamma_B(\vec{P}_{cm}^{\,F} - \vec{P}_{cm}^{\,I}) - (\vec{P}_r^{\,F} - \vec{P}_r^{\,I}) \quad , \quad (2.105b)$$

and the densities are those defined by Eqs. (2.100a) and (2.100b). The arguments in the one-body densities of the spectators indicate their role in absorbing recoil momenta in the interaction of the participating nucleon pair.

We can now write for the matrix elements of T(2) the expression

$$\langle T(2) \rangle = A\,B \int d\vec{p}\, d\vec{k} (\vec{p}\,'\vec{k}\,' | t | \vec{p}\vec{k})$$

$$\times \rho_A^{(1)}(\vec{p} - \gamma_A \vec{P}_{cm}^{\,I} - \vec{P}_r^{\,I}) \rho_B^{(1)}(\vec{k} - \gamma_B \vec{P}_{cm}^{\,I} + \vec{P}_r^{\,I}) \quad , \quad (2.106)$$

where we have taken advantage of the indistinguishability of the nucleons to obtain the result in terms of one-body densities independent of the particle labels. If we look at the limiting case of $A \to 1$ and $B \to 1$, we find that Eq. (2.106) reduces exactly to the two-body t-matrix for the free scattering of two nucleons.

The expression for T(2) given in Eq. (2.106) would also be obtained for the first order optical potential. It is noteworthy that from these results one immediately obtains the double-folded potential

(42). In conjunction with the development of Section E, it would be possible to develop the higher order correlation corrections to the double-folded potential.

III. DIRECT REACTION AMPLITUDE AND EIKONAL APPROXIMATION

In this chapter I am concerned with the formal development of the two-potential (30) form for the direct reaction transition amplitude and the eikonal approximation (43) thereof. In Section A, a number of operator identities are developed which are of general utility and which will facilitate the construction of the two-potential amplitude of Section B. Some brief comments on the properties of these identities are made. Sections C and D are concerned with the eikonal approximation to the transition amplitude and the nucleon-nucleon wave operators respectively.

A. Operator Identities

Let $\{H_j\}$ be a set of Hamiltonians governing the behavior of a system of particles under the influence of a set of particle interactions labeled by (j). For each member Hamiltonian H_j there exists an associated Green's function defined by

$$G_j(z) = (z - H_j)^{-1} \quad . \tag{3.1}$$

Where z is the complex parametric energy which, in the limit as z approached the on-shell energy of the system, carries the boundary conditions of the scattering process. The difference between two member Hamiltonians may be designated the interaction V_{jk}. This is defined by

$$V_{jk} = H_j - H_k \quad . \tag{3.2}$$

The interaction set $\{V_{jk}\}$ may be regarded as a set of perturbing potentials, which may be used to generate a perturbation expansion about a particular member Hamiltonian H_j.

The resolvent identity is defined as

$$a = b + b(b^{-1} - a^{-1})a \quad , \tag{3.3}$$

and implies the resolvent expressions for the Green's functions G_j and G_k. That is,

$$G_j = G_k + G_k(H_j - H_k)G_j \tag{3.4}$$

or equivalently

$$G_j = G_k + G_k V_{jk} G_j \quad . \tag{3.5}$$

Furthermore, the definitions Eq. (3.1) of the Green's functions can be used to show the following identity for the operator inverses $\{G_j^{-1}\}$.

$$G_j^{-1} = G_k^{-1} + H_k - H_j \tag{3.6}$$

and

$$G_j^{-1} = G_k^{-1} + V_{jk} \quad . \tag{3.7}$$

Consider a subset of the Hamiltonians $\{H_j\}$ defined to govern the system of particles partitioned into two bound clusters. Specifically let $\{h_j\}$ be the set of distinct noninteracting two-cluster Hamiltonians. For each noninteracting two-cluster Hamiltonian h_j there exists a set of two-cluster interactions denoted $\{v^{\ell}(j)\}$, where the index (j)

indicates a specific two-cluster partition and the index (ℓ) runs over all the possible cluster-cluster interaction terms. From these interactions we construct a set of interaction Hamiltonians, defined by

$$H_\ell(j) = h_j + v^\ell(j) \quad . \tag{3.8}$$

The Green's functions associated with h_j and $\{H_\ell(j)\}$ are denoted by g_j and $G_\ell(j)$ respectively.

The ket-vectors $|\phi(j\alpha)\rangle$ are defined to be the eigensolutions of channel (j) and state (α) of the eigenequation

$$[E(j\alpha) - h_j]|\phi(j\alpha)\rangle = 0 \quad . \tag{3.9}$$

Using the procedure of Chapter II, Section A leads to the construction of ket-vectors $|\chi_\ell^{(\pm)}(j\alpha)\rangle$, which are solutions to the eigenequation

$$[E(j\alpha) - H_\ell(j)]|\chi_\ell^{(\pm)}(j\alpha)\rangle = 0 \quad , \tag{3.10}$$

in terms of the noninteracting Hamiltonian ket-vector. That is,

$$|\chi_\ell^{(\pm)}(j\alpha)\rangle = |\phi(j\alpha)\rangle + G_\ell(j)\, v^\ell(j)|\phi(j\alpha)\rangle \quad . \tag{3.11}$$

This may be rewritten in terms of the Green's functions and the Green's function inverses as follows

$$\begin{aligned} |\chi_\ell^{(\pm)}(j\alpha)\rangle &= [1 + G_\ell(j)\, v^\ell(j)]|\phi(j\alpha)\rangle \\ &= G_\ell(j)[G_\ell^{-1}(j) + v^\ell(j)]|\phi(j\alpha)\rangle \quad . \end{aligned}$$

Apply the identity Eq. (3.7). Then

$$|\chi_{\ell}^{(\pm)}(j\alpha)\rangle = G_{\ell}(j)\, g_j^{-1}|\phi(j\alpha)\rangle \quad . \tag{3.12}$$

The inverse equation is readily seen to be

$$|\phi(j\alpha)\rangle = g_j\, G_{\ell}^{-1}(j)|\chi_{\ell}^{(\pm)}(j\alpha)\rangle \quad . \tag{3.13}$$

Using Eq. (3.13) for ϕ and ℓ' leads to

$$|\chi_{\ell}^{(\pm)}(j\alpha)\rangle = G_{\ell}(j)\, G_{\ell'}^{-1}(j)|\chi_{\ell'}^{(\pm)}(j\alpha)\rangle \quad . \tag{3.14}$$

The identity Eq. (3.14) describes the connection between two scattered waves evolving from the same two-cluster channel under the influence of different cluster interactions. Suppose that $v^{\ell}(j)$ is the 'exact' microscopic interaction between two nuclei, and that $v^{\ell'}(j)$ is some phenomenological optical potential. Then Eq. (3.14) describes the connection between the 'exact' scattering vector and the model vector. It provides a starting point for a study of the systematic departure of the many-body wave vector from the more readily calculated model wave vector. The operator $G_{\ell}(j)\, G_{\ell'}^{-1}(j)$ may be expanded using the spectator expansion described in Chapter II. As such this provides a ready framework for discussing convergence and the appositeness of various approximations.

B. Two Potential Form of Transition Matrix

The theory of the scattering matrix S (Chapter II, Section A) describes the on-shell transition matrix τ by the two equivalent expressions,

$$\tau(i_\alpha \to f_\beta) = \langle \Psi_f^{(-)}(\beta) | V^i | \phi_i(\alpha) \rangle \tag{3.15a}$$

and

$$\tau(i_\alpha \to f_\beta) = \langle \phi_f(\beta) | V^f | \Psi_i^{(+)}(\alpha) \rangle \quad . \tag{3.15b}$$

Here the indices (i) and (f) specify the entrance and exit channels respectively, whereas (α) and (β) are state labels in the respective channels.

Consider the action of an entrance channel perturbation (denoted by U^i) upon the eigenket $|\phi_i(\alpha)\rangle$. From the identity Eqs. (3.12) and (3.13) it is evident that

$$|\phi_i(\alpha)\rangle = G_i(G_i^{-1} - U^i) |\chi_i^{(+)}(\alpha)\rangle \quad . \tag{3.16}$$

and

$$\langle \Psi_f^{(-)}(\beta)| = \langle \phi_f(\beta) | G_f^{-1} G \quad . \tag{3.17}$$

Applying the Eqs. (3.16) and (3.17) to Eq. (3.15a), we obtain for the transition matrix element the expression

$$\tau(i_\alpha \to f_\beta) = \langle \phi_f(\beta) | G_f^{-1} G[V^i][G_i(G_i^{-1} - U^i)] | \chi_i^{(+)}(\alpha) \rangle . \tag{3.18}$$

The resolvent identity Eq. (3.5) applied to G and G_i leads to

$$G = G_i + G_i V^i G = G_i + G V^i G_i \quad . \tag{3.19}$$

Thus

$$G_i \, V^i \, G = G \, V^i \, G_i \quad , \tag{3.20}$$

and Eq. (3.18) becomes

$$\tau(i_\alpha \to f_\beta) = \langle \phi_f(\beta) \, G_f^{-1} \, G_i \, V^i \, G(G_i^{-1} - U^i) | \chi_i^{(+)}(\alpha) \rangle \quad . \tag{3.21}$$

From the resolvent inverse Eq. (3.7), note that

$$G^{-1} = G_i^{-1} - V^i \quad , \tag{3.22}$$

this implies that

$$\begin{aligned} \tau(i_\alpha \to f_\beta) &= \langle \phi_f(\beta) | G_f^{-1} \, G_i \, V^i \, G(G^{-1} + V^i - U^i) | \chi_i^{(+)}(\alpha) \rangle \\ &= \langle \phi_f(\beta) | G_f^{-1} \, G_i \, V^i | \chi_i^{(+)}(\alpha) \rangle + \langle \phi_f(\beta) | G_f^{-1} \, G_i \, V^i \, G(V^i - U^i) \\ &\quad \times | \chi_i^{(+)}(\alpha) \rangle \quad . \end{aligned} \tag{3.23}$$

Applying Eq. (3.19) this becomes

$$\begin{aligned} \tau(i_\alpha \to f_\beta) &= \langle \phi_f(\beta) | G_f^{-1} \, G_i \, V^i | \chi_i^{(+)}(\alpha) \rangle + \langle \phi_f(\beta) | G_f^{-1} (G - G_i) \\ &\quad \times (V^i - U^i) | \chi_i^{(+)}(\alpha) \rangle \\ &= \langle \phi_f(\beta) | G_f^{-1} \, G(V^i - U^i) | \chi_i^{(+)}(\alpha) \rangle + \langle \phi_f(\beta) \, G_f^{-1} \, G_i \, U^i \\ &\quad \times | \chi_i^{(+)}(\alpha) \rangle \quad , \end{aligned} \tag{3.24}$$

or equivalently

$$\tau(i_\alpha \to f_\beta) = \langle \psi_f^{(-)}(\beta) | (V^i - U^i) | \chi_i^{(+)}(\alpha) \rangle + \langle \phi_f(\beta) | G_f^{-1} G_i U^i | \chi_i^{(+)}(\alpha) \rangle \quad . \tag{3.25}$$

Equation (3.25) is the form alluded to in the literature as the two-potential formula (30).

It has been incorrectly asserted by Goldberger and Watson (44) that the second term of Eq. (3.25) vanishes identically, regardless of the interaction U^i. If the interaction U^i is an optical potential connecting only elastic and inelastic scattered states of the incident channel, then the so-called Lippmann's identity (36) obtains and this term is indeed zero. However, for a general many-body scattering interaction this is not the case and this term may connect breakup states of the incident channel with the two-cluster exit channel. Fortunately, for most physical processes in which the two-potential formula has been applied, the first term appears dominant, and secondly the approximations made in using distorted wave approximations often explicitly force the second term to zero.

Suppose we consider that class of direct reactions described as two-cluster rearrangement collisions. Schematically such processes may be represented by

$$A + (BC) \to (AC) + B \quad , \tag{3.26}$$

where (BC) and (AC) are bound states of the subclusters B, C and A, C respectively. In the entrance and exit channels A, B, (BC), and (AC) are readily identified as distinct bound states of more elementary

constituents. The reaction (see Fig. 5) describes the transfer of the subcluster C from a bound state (BC) to a bound state (AC).

The Hamiltonians for A, B, and C are h_A, h_B, and h_C. The total noninteracting Hamiltonian H_o is defined

$$H_o = h_A + h_B + h_C \quad . \tag{3.27}$$

The subcluster interactions which form the bound states (BC) and (AC) are V_{BC} and V_{AC}. The total interaction between the clusters A, B, and C is

$$V^o = V_{AB} + V_{BC} + V_{AC} \quad . \tag{3.28}$$

The channel Hamiltonians for the reaction Eq. (3.26) are

$$H_i = H_o + V_{BC} = H_o + V_i \tag{3.29}$$

and

$$H_f = H_o + V_{AC} = H_o + V_f \quad . \tag{3.30}$$

The channel interactions become

$$V^i = V^o - V_i = V_{AB} + V_{AC} \tag{3.31}$$

and

$$V^f = V^o - V_f = V_{AB} + V_{BC} \quad . \tag{3.32}$$

The total Hamiltonian H_T may be described as

$$H_T = H_a + V^a \quad , \quad a \in (o, i, f) \quad . \tag{3.33}$$

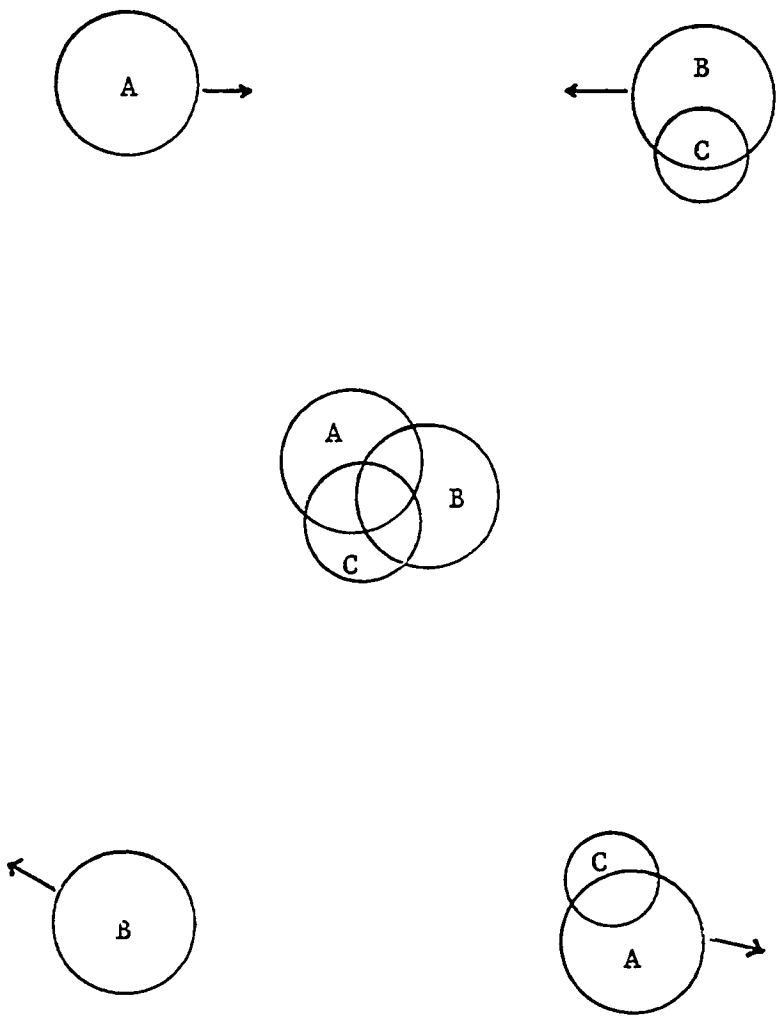


Figure 5. Two-cluster rearrangement collision

The corresponding propagators are

$$G_a = (Z - H_a)^{-1} \quad , \quad a \in (i, f, T) \quad . \tag{3.34}$$

Consider the two-potential formula as described by Eq. (3.25); letting $U^i = V_{AB}$, this becomes

$$\tau(i_\alpha \to f_\beta) = \langle \Psi_f^{(-)}(\beta)\ V_{AC} | \chi_i^{(+)}(\alpha) \rangle + \langle \phi_f(\beta) | G_f^{-1}\ G_i\ V_{AB} | \chi_i^{(+)}(\alpha) \rangle \quad . \tag{3.35}$$

The contributions from the second term of Eq. (3.35) depend upon the exact nature of the rearrangement channel and energy regime of interest. For incident energies below some critical energy E_c, no breakup states of the incident channel may exist.

Note that the bra $\langle \Psi_f^{(-)}(\beta)|$ and ket $|\chi_i^{(+)}(\alpha)\rangle$ may be written using the identities of Section A as

$$\langle \Psi_f^{(-)}(\beta)| = \langle \phi_f(\beta)| + \langle \Psi_f^{(-)}(\beta)| (V_{AB} + V_{BC}) \frac{1}{E - in - H_f} \tag{3.36}$$

and

$$|\chi_i^{(+)}(\alpha)\rangle = |\phi_i(\alpha)\rangle + \frac{1}{E - in - H_i} V_{AB} |\chi_i^{(+)}(\alpha)\rangle \quad . \tag{3.37}$$

Equations (3.36) and (3.37) display the integral solution to the entrance and exit wavefunctions in the iterative form. In the next section, the eikonal approximation to these wavefunctions is developed.

C. The Eikonal Approximation

From Chapter II, we know that any operator or integral quantity may be expanded about an arbitrary set of functions. As a matter of course, one generally selects a set such that convergence to the 'real' or model problem is enhanced. In the case of intermediate energies the eikonal (or high energy approximation) (43) has been used with some success in elastic and inelastic scattering (45-48). Tekou (49) has applied it to the formulation of the pickup reaction. My approach in Sections C and D follows closely his development, although certain points of emphasis are different.

Consider the transition amplitude for the neutron pickup reaction, in which a proton incident upon nucleus A picks up a neutron to form a deuteron leaving a residual nucleus B. Schematically this is

$$p + A \rightarrow d + B \quad . \tag{3.38}$$

In the following development I shall only consider the first term in the two-potential formula Eq. (3.35), since the second term does not contribute to the direct reaction amplitude but may be of some interest in higher order processes.

Let Φ_{in} and Φ_{out} represent channel solutions of the entrance and exit channel eigenequations. In detail, the unsymmetrized wavefunctions may be written as

$$\Phi_{in} = \left(\frac{1}{2\pi}\right)^3 e^{i\vec{k}_p\cdot\vec{r}_p + i\vec{k}_A\cdot\vec{R}_A} \chi^p_{S_p\mu_p} \sum_{L_A m_A, S_A\mu_A} \langle J_A M_A | L_A m_A S_A \mu_A \rangle$$

$$x\ U^{A}_{L_A m_A}\ X^{A}_{S_A \mu_A} \tag{3.39}$$

and

$$\Phi_{out} = \left(\frac{1}{2\pi}\right)^3 e^{i\vec{k}_d\cdot\vec{r}_d + i\vec{k}_B\cdot\vec{R}_B} \sum_{L_B m_B, S_B \mu_B} \langle J_B M_B | L_B m_B S_B \mu_B \rangle U^{B}_{L_B m_B}\ X^{B}_{S_B \mu_B}$$

$$x \sum_{L_d m_d, S_d \mu_d} \langle J_d M_d | L_d m_d S_d \mu_d \rangle U^{d}_{L_d m_d}\ X^{d}_{S_d \mu_d} \quad . \tag{3.40}$$

Here the 'X' is the spin function, the 'U' is the space wavefunction, the '<|>' is the standard Clebsch-Gordon coupling constants (35), and the '$\vec{k}$' is the momentum vector of the various particle clusters.

We need only consider the matrix elements of

$$T_{pd} = \langle \Psi^{(-)}_{out}(\beta) | V_{pn} | \bar{\Psi}^{(+)}_{in}(\alpha) \rangle \quad . \tag{3.41}$$

The stationary states $\Psi^{(-)}_{out}(\beta)$ and $\bar{\Psi}^{(+)}_{in}(\alpha)$ are those evolved from $\Phi_{out}(\beta)$ and $\Phi_{in}(\alpha)$ according to Eqs. (3.36) and (3.37) respectively. The solutions of these equations cannot be obtained exactly for this example when realistic nucleon potentials are employed.

Under the assumption of sufficiently high entrance and exiting energy (31) of the proton and deuteron respectively, the wavefunctions in the eikonal approximation may be written as

$$\bar{\Psi}^{(+)}_{in} = \Phi_{in}(\vec{r}_p, \vec{r}_n, \vec{r}_1, \vec{r}_2 \cdots \vec{r}_B) e^{i\phi^{+}_{pB}(\vec{r}_p, \vec{r}_1, \vec{r}_2, \cdots \vec{r}_B)} \tag{3.42}$$

and

$$\psi^{(-)}_{out} = \Phi_{out}(\vec{r}_p,\vec{r}_n,\vec{r}_1,\vec{r}_2 \cdots \vec{r}_B)e^{-i\phi^-_{dB}(\vec{r}_p,\vec{r}_n,\vec{r}_1,\vec{r}_2, \cdots \vec{r}_B)} \quad .(3.43)$$

The functions ϕ^+_{pB} and ϕ^-_{dB} are the eikonal phase shifts and are defined by the expressions

$$\phi^+_{pB} = \sum_{j\epsilon B} \chi^+_{pj}(\vec{r}_p - \vec{r}_j) \quad (3.44)$$

and

$$\phi^-_{dB} = \sum_{j\epsilon B} \{\chi^-_{pj}(\vec{r}_p - \vec{r}_j) + \chi^-_{mj}(\vec{r}_m - \vec{r}_j) \quad , \quad (3.45)$$

with

$$\chi^+_{j\ell}(r) = -\frac{1}{\hbar v_{j\ell}} \int_{-\infty}^{z} V_{j\ell}(b,z')dz' = -\frac{1}{\hbar v_{pB}} \int_{-\infty}^{z} V_{j\ell}(b,z')dz' \quad (3.46)$$

and

$$\chi^-_{j\ell}(r) = -\frac{1}{\hbar v_{j\ell}} \int_{z}^{\infty} V_{j\ell}(b,z')dz' = -\frac{1}{\hbar v_{dB}} \int_{z}^{\infty} V_{j\ell}(b,z')dz' \quad (3.47)$$

In Eqs. (3.46) and (3.47), $v_{j\ell}$ is the relative velocity of particles j and ℓ. At high incident and exiting velocity $v_{j\ell}$ may be reasonably replaced by v_{pB} and v_{dB} respectively.

The amplitude of the neutron pickup reaction Eq. (3.41) with full spin and angular coupling is written as

$$T_{pd} = T(J_A M_A \; S_p \mu_p | J_B M_B \; J_d M_d)$$

$$= \sum \langle J_A M_A \; L_A m_A \; S_A \mu_A \rangle \langle J_B M_B \; L_B m_B \; S_B \mu_B \rangle \langle J_d M_d \; L_d m_d \; S_d \mu_d \rangle$$

sum over repeated indices

$$x \langle S_d \mu_d \; S_B \mu_B | T(L_A m_A; \; L_B m_B \; L_d m_d) | S_A \mu_A \; S_p \mu_p \rangle \quad , \tag{3.48}$$

where the matrix element $T(L_A m_A; \; L_B m_B \; L_d m_d)$ is defined by

$$T(L_A m_A; \; L_B m_B \; L_d m_d) \equiv \left(\frac{1}{2\pi}\right)^3 \int d^3 r_p \; d^3 r_n \; e^{i\vec{Q}_p \cdot \vec{r}_p + i\vec{Q}_n \cdot \vec{r}_n}$$

$$x \; d_{L_d m_d}(\vec{r}_p - \vec{r}_n) N^{L_B m_B}_{L_A m_A}(\vec{r}_p, \vec{r}_n) \quad . \tag{3.49}$$

$$\vec{Q}_p = \vec{k}_p - \tfrac{1}{2}\vec{k}_d \tag{3.50}$$

$$\vec{Q}_n = -\tfrac{1}{2}\vec{k}_d - 1/A \; \vec{k}_p \quad . \tag{3.51}$$

In Eq. (3.49) the quantity '$d_{L_d m_d}$' is the deuteron vertex function defined in Appendix C and contains the information about the deuteron wavefunction and the proton-neutron interaction. The function 'N' contains the neutron overlap wavefunction between the incident A-particle nucleus and the residual B-particle nucleus as well as the distortion factors arising from the interaction of the B-spectators in both the entrance and exit channels. In terms of the wavefunction U^B and U^A, this function is

$$N^{L_B m_B}_{L_A m_A}(\vec{r}_p, \vec{r}_n) = \int d^3 r_1 \cdots d^3 r_B \; U^B_{L_B m_B}(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_B)$$

$$x \; W(\vec{r}_p, \vec{r}_n, \vec{r}_1, \vec{r}_2 \cdots \vec{r}_B) U^A_{L_A m_A}(\vec{r}_n, \vec{r}_1, \vec{r}_2 \cdots \vec{r}_B) \quad . \tag{3.52}$$

and W, the distortion function, is written in terms of the eikonal phase shifts as

$$W(\vec{r}_p,\vec{r}_n, \cdots \vec{r}_B) = \exp[i\phi^{+}_{pB}(\vec{r}_p,\vec{r}_1, \cdots \vec{r}_B) + i\phi^{-}_{dB}(\vec{r}_p,\vec{r}_n,\vec{r}_1, \cdots \vec{r}_B)] \quad . \tag{3.53}$$

If we now define the individual nucleon distortion functions $w^{\pm}(r)$ by

$$w^{\pm}_{j\ell}(\vec{r}_{j\ell}) \equiv 1 - \exp(i\chi^{\pm}_j(\vec{r}_{j\ell})) \quad , \tag{3.54}$$

then Eq. (3.53) may be written as

$$W = \prod_{j\varepsilon B} [1 + \bar{w}_j(\vec{r}_j,\vec{r}_p,\vec{r}_n)] \quad , \tag{3.55}$$

where

$$\bar{w}_j(\vec{r}_j,\vec{r}_p,\vec{r}_n) = [1 - w^{+}_{pj}(\vec{r}_{pj})][1 - w^{-}_{pj}(\vec{r}_{pj})][1 - w^{-}_{nj}(\vec{r}_{nj})] - 1 \quad . \tag{3.56}$$

By inserting a complete set of wavefunctions of the residual target B, the function N may be written as the product of the pickup neutron wavefunction U(r) and the integrated distortion factor $F(r_p,r_n)$. That is,

$$N^{L_B m_B}_{L_A m_A}(\vec{r}_p,\vec{r}_n) = \sum_{\lambda_B \nu_B} U^{\lambda_B \nu_B}_{L_A m_B}(\vec{r}_n)\, F^{L_B m_B}_{\lambda_B \nu_B}(\vec{r}_p,\vec{r}_n) \quad , \tag{3.57}$$

with

$$U^{\lambda_B \nu_B}_{L_B m_B}(\vec{r}_n) = \int d^3r_1 \cdots d^3r_B\, U^{B*}_{\lambda_B \nu_B}\, U^{A}_{L_A m_A} \tag{3.58}$$

and

$$F^{L_B m_B}_{\lambda_B \nu_B}(\vec{r}_p, \vec{r}_n) = \int d^3r_1 \cdots d^3r_B \; U^{B*}_{L_B m_B} \; W \; U^{B}_{\lambda_B \nu_B} \quad . \tag{3.59}$$

It makes apparent how one can explicitly include multistep processes. Intermediate states may easily be included in the incident channel as well. Naturally, an extensive proliferation of such states will result in a computationally intensive problem. Fortunately, the physical process of interest appears to be dominated by diagonal elements. A direct nuclear reaction is one dominated by at most a few states coupled significantly to the initial and final states.

One estimate of the contribution of each intermediate state neutron wavefunction of nucleus B may be obtained by the computation of the spectroscopic factors $|\beta(\lambda_B \nu_B \; L_A m_A)|^2$.

$$|\beta(\lambda_B \nu_B \; L_A m_A)|^2 = \int d^2r |U^{\lambda_B \nu_B}_{L_A m_A}(r)|^2 \quad . \tag{3.60}$$

D. The Nucleon-Nucleon Distortion Functions

The evaluation of the distortion factor F which represents the perturbing effect of the B-spectator particles on the incoming and outgoing waves, depends in detail upon the distortion function W Eq. (3.55). As we can see from Eqs. (3.54), (3.55) and (3.56), this ultimately requires some knowledge of the eikonal nucleon-nucleon wave shifts.

The nucleon-nucleon eikonal phase shift functions $\chi^{\pm}(r)$ are related to the elastic nucleon-nucleon scattering amplitude $A(\vec{q})$. Using the

eikonal propagator the relation is

$$A(\vec{q}) = -\mu/2\pi \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(\vec{r})\, e^{i\chi^+(r)} \quad , \tag{3.61}$$

where μ is the reduced nucleon-nucleon mass.

In order to demonstrate the connection between the scattering amplitude $A(\vec{q})$ and nucleon-nucleon distortion function $w^{\pm}(\vec{r})$, note that Eqs. (3.54) may also be written in integral form as

$$w^+(\vec{r}) = i/\hbar v \int_{-\infty}^{z} V(b,z')e^{i\chi^+(b,z')}dz' \tag{3.62a}$$

and

$$w^-(\vec{r}) = i/\hbar v \int_{z}^{\infty} V(b,z')e^{i\chi^-(b,z')}dz' \quad . \tag{3.62b}$$

Taking the Fourier transform of $A(\vec{q})$, then yields for $w^{\pm}(\vec{r})$ the expressions

$$w^+(\vec{r}) = \frac{1}{(2\pi)^2 ik} \int d^3q\, A(\vec{q}_t,q_z)e^{-\vec{q}_t\cdot\vec{b}} \int_{-\infty}^{Z} e^{-iq_z\cdot z'} dz' \tag{3.63a}$$

and

$$w^-(\vec{r}) = \frac{1}{(2\pi)^2 ik} \int d^3q\, A(\vec{q}_t,q_z)e^{-i\vec{q}_t\cdot\vec{b}} \int_{z}^{\infty} e^{-iq_z\cdot z'} dz' \quad , \tag{3.63b}$$

where $\vec{q}_t$ and q_z are the transverse and longitudinal momentum transfers respectively.

The nucleon-nucleon amplitude $A(\vec{q})$ may be expanded in a Taylor series about $q_z = 0$. Thus we obtain the expression

$$A(\vec{q}) = \sum_{n=0}^{\infty} \frac{(q_z)^n}{n!} A^{(n)}(\vec{q}_t, 0) \quad , \tag{3.64}$$

where

$$A^{(n)}(\vec{q}_t, 0) = (\partial/\partial q_z)^n A(q)\Big|_{q_z=0} \quad . \tag{3.65}$$

Substitution of Eq. (3.64) into Eqs. (3.63a) and (3.63b) yields the relations

$$w^{\pm}(r) = \frac{1}{2\pi ik} \sum_{n=0} \frac{(i)^n}{n!} \phi_n^{\pm}(z) \bar{A}^{(n)}(b) \quad , \tag{3.66}$$

$$= \sum_{n=0} w_n^{\pm}(r) \quad , \tag{3.67}$$

where

$$\Phi_n^{+}(z) = \int_{-\infty}^{z} \delta^{(n)}(z')dz' \tag{3.68a}$$

and

$$\Phi_n^{-}(z) = \int_{z}^{\infty} \delta^{(n)}(z')dz' \quad . \tag{3.68b}$$

Here $\delta(z)$ is the usual Dirac delta function, and the derivatives $\delta^{(n)}(z)$ are defined by

$$\delta^{(n)}(z) = (d/dz)^n \delta(z) \quad . \tag{3.69}$$

The properties of the functions $\Phi_{(n)}^{\pm}(z)$ deserve some comment. For $n = 0$, they become

$$\Phi_0^+(z) = \theta^+(z) = \begin{cases} 1 & z > 0 \\ 1/2 & z = 0 \\ 0 & z < 0 \end{cases} \tag{3.70a}$$

$$\Phi_0^-(z) = \theta^-(z) = \begin{cases} 0 & z > 0 \\ 1/2 & z = 0 \\ 1 & z < 0 \end{cases} . \tag{3.70b}$$

Note also the reflection of $\Phi^-_{(n)}(z)$.

$$\Phi^-_{(n)}(z) = \int_z^\infty \delta^{(n)}(z')dz'$$

$$= \int_{-\infty}^{-z} \delta^{(n)}(-z')dz'$$

$$\Phi^-_{(n)}(z) = -\int_{-\infty}^{z} \delta^{(n)}(z')dz' \quad ; \quad n = 1, 2, 3, \cdots$$

$$\Phi^-_{(n)}(z) = -\Phi^+_{(n)}(z) \quad ; \quad n = 1, 2, 3, \cdots \quad . \tag{3.71}$$

Equation (3.71) is the reflection identity for $\Phi^{(-)}_{(n)}$.

Consider a typical representation of the spin-isospin averaged nucleon-nucleon scattering amplitude to be the parametrization

$$A(q) = \frac{ik}{4\pi} \sigma_{NN}^{TOT}(1 - i\alpha^{NN})e^{-\beta^2 q^2/4} \quad . \tag{3.72}$$

In Eq. (3.72), k is the relative momentum, σ_{NN}^{TOT} is total nucleon-nucleon cross section at the interaction energy, α^{NN} is ratio of real to imaginary strengths of the forward amplitude, and β corresponds to a measure of the nucleon-nucleon interaction range. In general k, σ_{NN}^{TOT},

α^{NN}, and β will be different for the incoming and outgoing waves. The distinction will, however, be ignored for the present.

The parametrization Eq. (3.72) yields, when inserted in Eq. (3.66) for the $w_n^{\pm}(\vec{r})$,

$$w_n^{\pm}(r) = \frac{(+i)^n}{n!} \Gamma(b)\, \Phi_n^{\pm}(z)(-\beta/2)^n H_n(0) \quad , \tag{3.73}$$

where

$$\Gamma(b) = \frac{1}{2\pi\, \beta^2}\, \sigma_{NN}^{TOT}(1 - i\alpha^{NN})e^{-b^2/\beta^2} \quad , \tag{3.74}$$

and the $H_n(0)$ are the Hermite polynomials (50) evaluated at zero. The $H_n(0)$ satisfy the recurrence equation

$$H_{n+2}(0) = (-2)(n + 1)H_n(0) \quad . \tag{3.75}$$

The initial values $H_o = 1$ and $H_1 = 0$, then imply that in $w^{\pm}(\vec{r})$ all n = odd integer terms vanish, and furthermore that

$$w_o^{\pm}(\vec{r}) = \Gamma(b)\theta^{\pm}(z) \tag{3.76}$$

$$w_n^{\pm}(\vec{r}) = \pm\, \Gamma(b)\delta^{(n-1)}(z)\, \frac{(\beta/\sqrt{2})^n}{n!}\, [(n - 1)!!] \tag{3.77}$$

$$n = 2,\ 4,\ 6,$$

Define now the quantities $\tilde{w}^{\pm}(z)$, such that

$$w^{\pm}(\vec{r}) = \Gamma(b)\, \tilde{w}^{\pm}(z) \quad . \tag{3.78}$$

We can immediately write down the expression for $\tilde{w}^{\pm}(z)$.

$$\tilde{w}^{\pm}(z) = \theta^{\pm}(z) \pm \sum_{n=2 \text{ (even integers)}} \delta^{(n-1)}(z) \, (\beta/\sqrt{2})^n \, \frac{(n-1)!!}{n!} \qquad (3.79)$$

The nucleon-nucleon distortion function $w^{\pm}(\vec{r})$ Eq. (3.79) is often truncated at first term. The effects of such truncation on an arbitrary density function $\rho(r)$ are investigated in Chapter IV. Typical Glauber multiple scattering approximations (17,18,45) are a result of such a truncation in the application of the eikonal procedure to elastic and inelastic scattering. Different levels of truncation of Eq. (3.79) result in a specific family of spectator expansions.

IV. SENSITIVITIES OF THE REACTION CROSS SECTION IN AN EIKONAL MODEL

A. Convergence of the Nucleon-Nucleon Distortion Functions

The nucleon-nucleon distortion functions $w^{\pm}(\vec{r})$ appear only in integrals over nuclear density functions. Consider a model density function $\rho(r)$, which is parametrized by a normalized gaussian distribution of the form

$$\rho(r) = \frac{1}{R_o^3\, \pi^{3/2}}\, e^{-r^2/R_o^2} \quad . \tag{4.1}$$

Here the RMS radius is $\sqrt{3}/2\ R_o$ and R_o is a measure of the nuclear size.

Define a function $T(\vec{r})$ by

$$T(\vec{r}) \equiv \int d^3r'\ \rho(r')\ w^{+}(\vec{r} - \vec{r}') \quad , \tag{4.2}$$

or alternatively by

$$T(\vec{r}) = \sum_{n=0}^{\infty} T_n(\vec{r}) \quad , \tag{4.3}$$

where by using Eq. (3.67), the $T_n(\vec{r})$ can be shown to equal

$$T_n(\vec{r}) = \int d^3r'\ \rho(r')\ w_n^{+}(\vec{r} - \vec{r}') \quad . \tag{4.4}$$

Substitution of Eqs. (3.76) and (3.77) into Eq. (4.4) then gives

$$T_o(\vec{r}) = t(\vec{b}) \begin{cases} 1 - \Gamma(\tfrac{1}{2}, z^2/R_o^2)/2\pi^{\frac{1}{2}} & z > 0 \\ \Gamma(\tfrac{1}{2}, a^2/R_o^2)/2\pi^{\frac{1}{2}} & z < 0 \end{cases} \tag{4.5}$$

and

$$T_n(\vec{r}) = t(\vec{b}) \{ - \frac{(n-1)!!}{n!} (\beta/\sqrt{2}\, R_o)^n h_{n-1}(Z/R_o)\} \quad , \qquad (4.6)$$

for n = 2, 4, 6, ···, and

$$T_n(\vec{r}) = 0 \quad , \quad \text{for } n = 1, 3, 5, \cdots \quad ; \qquad (4.7)$$

the function $\Gamma(\frac{1}{2},x^2)$ is an incomplete gamma function (50) defined by the equation

$$\Gamma(\tfrac{1}{2},x_o^2) = 2 \int_{x_o}^{\infty} e^{-t^2}\, dt \quad , \quad x_o \geq 0 \quad , \qquad (4.8)$$

and the transverse function $t(\vec{b})$ is defined by

$$t(\vec{b}) = \sigma_{NN}^{TOT}(1 - i\alpha_{NN})/(\beta^2 + R_o^2)/2\pi \quad . \qquad (4.9)$$

The functions $h_{n-1}(x)$ are the Gaussian weighted Hermite polynomials

$$h_n(x) = H_n(x)\, e^{-x^2} \quad , \qquad (4.10)$$

and satisfy the recursion equation

$$h_{n+1}(x) = 2x\, h_n(x) - 2n\, h_{n-1}(x) \quad ; \qquad (4.11)$$

the initial values of $H_n(x)$ are $H_o = 1$ and $H_1 = 2x$.

From the results of Eqs. (4.5), (4.6) and (4.7) it is evident that $T(\vec{r})$ is factorable into a function of Z, and the function t(b) of the transverse coordinate. We may write

$$T(\vec{r}) = t(b)\, z(Z) \quad . \qquad (4.12)$$

The function $z(Z)$ is defined for $Z > 0$ as

$$z(Z) = 1 - \Gamma(\tfrac{1}{2}, Z^2/R_o^2) - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!} \left(\frac{\beta^2}{2R_o^2} \right)^n h_{2n-1}(Z/R_o) \quad , \quad (4.13a)$$

and for $Z < 0$ as

$$z(Z) = \Gamma(\tfrac{1}{2}, Z^2/R_o^2) - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!} \left(\frac{\beta^2}{2R_o^2} \right)^n h_{2n-1}(Z/R_o) \quad . \quad (4.13b)$$

A careful examination shows that the function $(z(Z) - \frac{1}{2})$ is anti-symmetric about $Z = 0$. Figure 6 shows the characteristic behavior of $z(Z)$ for $\beta = 1.4$ and $R_o = 6.0$.

The convergence of $z(Z)$ may be investigated by an examination of the asymptotic behavior of the n-th term S_n of the series for fixed $x = Z(R_o)$.

$$S_n = (\beta^2/2R_o^2)^n \frac{(2n-1)!!}{(2n)!} h_{2n-1}(x) \quad . \quad (4.14)$$

From Abramowitz and Stegun (50), I obtain the asymptotic properties of the Hermite polynomials $H_n(x)$. Thus the asymptotic behavior of $h_n(x)$ may be written as

$$h_n(x) \simeq \left(\frac{x}{\pi}\right)^{\frac{1}{2}} 2^n e^{-x^2/2} \Gamma\left(\frac{n+1}{2} \right) \cos(x\sqrt{1+2n} - n\pi/2)$$

$$\cdot \left[1 + \mathcal{O}(|n + \tfrac{1}{2}|^{-\frac{1}{2}})\right] \quad . \quad (4.15)$$

Substitution of Eq. (4.15) into Eq. (4.14) yields after simplification

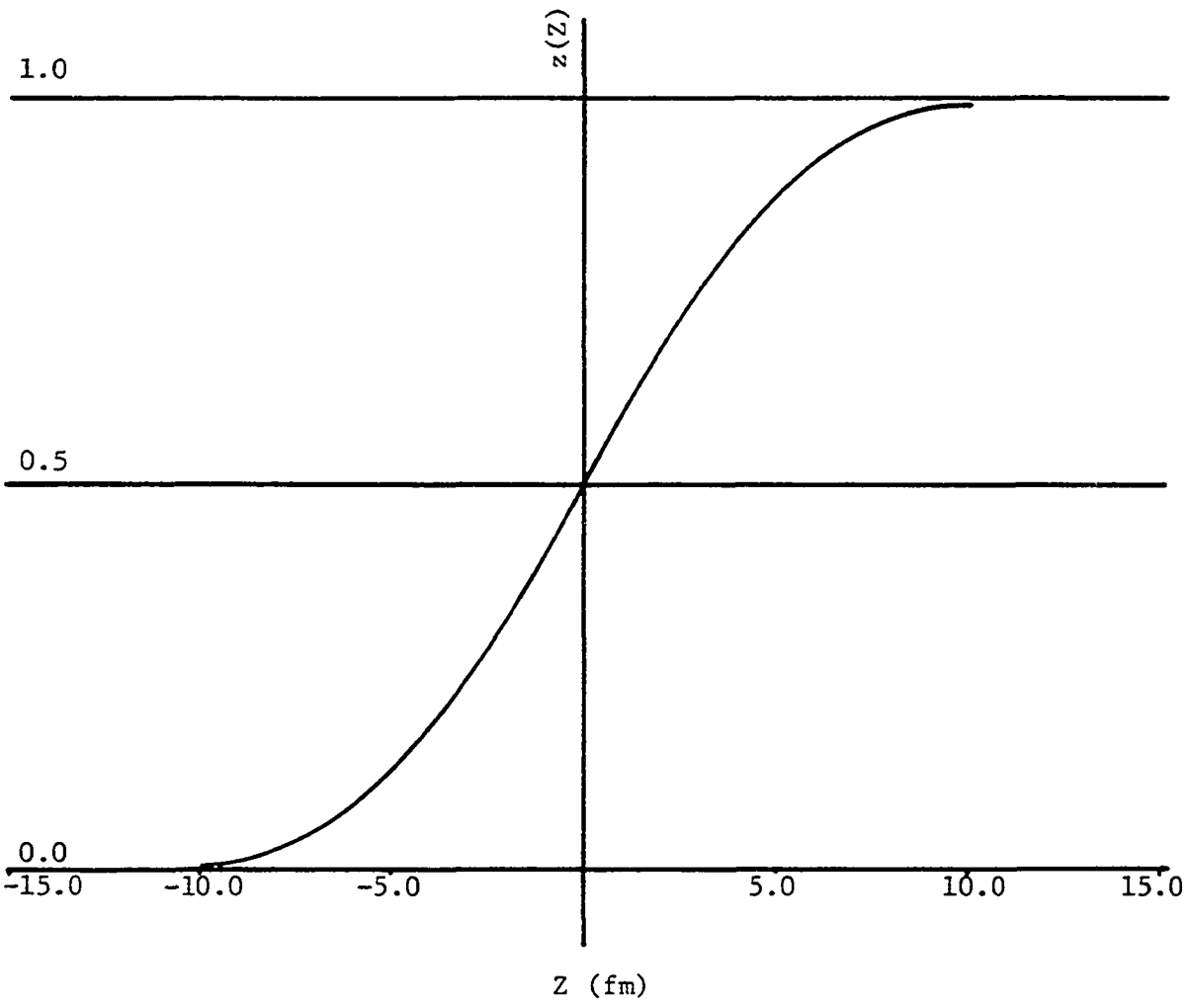


Figure 6. Converged function z(Z) for β = 1.4 fm and R_o = 6.0 fm

$$S_n \simeq \frac{(\beta^2/R_o^2)^n}{n} \left(\frac{x}{\pi}\right)^{\frac{1}{2}} e^{-x^2/2}(-)^n \sin(x\sqrt{4n-1})\left[1 + \mathcal{O}(|2n-\tfrac{1}{2}|^{-\frac{1}{2}})\right] . \quad (4.16)$$

Then we notice that

$$|S_n| \leq S_n^o \quad , \quad (4.17)$$

where

$$S_n^o \equiv \frac{(\beta^2/R_o^2)^n}{n} \left(\frac{x}{\pi}\right)^{\frac{1}{2}} e^{-x^2/2}\left[1 + a/(2n-\tfrac{1}{2})\right] \quad (4.18)$$

for some finite value a. The series F_M^o defined as

$$F_M^o \equiv \sum_{n=M}^{\infty} S_n^o \quad , \quad (4.19)$$

for some $M < \infty$ is seen to converge absolutely for all x, provided

$$|\beta/R_o| < 1 \quad , \quad (4.20)$$

by comparison of Eq. (4.19) with the geometric series

$$F_g = \sum_{n=0}^{\infty} r^n < \infty \quad , \quad \text{for } |r| < 1 \quad . \quad (4.21)$$

Since the series F_M^o dominates the series F_M, the function z(Z) converges absolutely for $|\beta/R_o| < 1$. Physically this restriction is roughly equivalent to a statement about the relative ranges of the nucleon-nucleon interaction and the size of the nuclear bound system.

For intermediate energies Vary and Dover (42) have found that for the parametrization of the scattering amplitude $A(\vec{q})$ Eq. (3.72) the

range β of the nucleon-nucleon interaction is roughly $1.2 \sim 1.6$ fm. For real nuclei the nuclear radius runs over a range of $1.9 \sim 5.5$ fm from helium through lead. Thus the function $z(Z)$ is well behaved even for small nuclear systems.

In Figure 7 the behavior of $z(Z)$ as a function of R_o is displayed. Note the smooth transition from the average value of $z(0) = 0.5$ to the maximum of 1.0. As a function of R_o the function $z(Z)$ appears to attain approximately 90% of full value for $z(R_o)$.

A question of considerable importance for real densities is the rate of convergence of the series for $z(Z)$ as a function of the nuclear size. In Fig. 8 I compare the results of using $w_o^+(Z)$, $(w_o^+(Z) + w_2^+(Z))$, and $w^+(Z)$ to obtain the function $z(Z)$. Notice in Fig. 8 that for the worst possible rate of convergence of $z(Z)$, i.e., (R_o small), the improvement between the zero-th order result and the second order result is not particularly great. A better way of viewing the difference is to consider the ratio of zero-th and second order terms to the converged result. From Fig. 9 we can see that the maximum discrepancy in the zero-th order case is $\sim$5% whereas for the second order case it is $\sim$3% for $R_o = 2$ fm. Furthermore as R_o increases to 4 fm the maximum discrepancy is of the order of $\sim$1.6% and even smaller for $R_o = 6$ fm. Our conclusion is that within the accuracy of the parametrization of the scattering amplitude, the function $z(Z)$ is well described by retaining only the first term. This is equivalent to approximating the distortion functions $w^{\pm}(\vec{r})$ by the first term alone, i.e.,

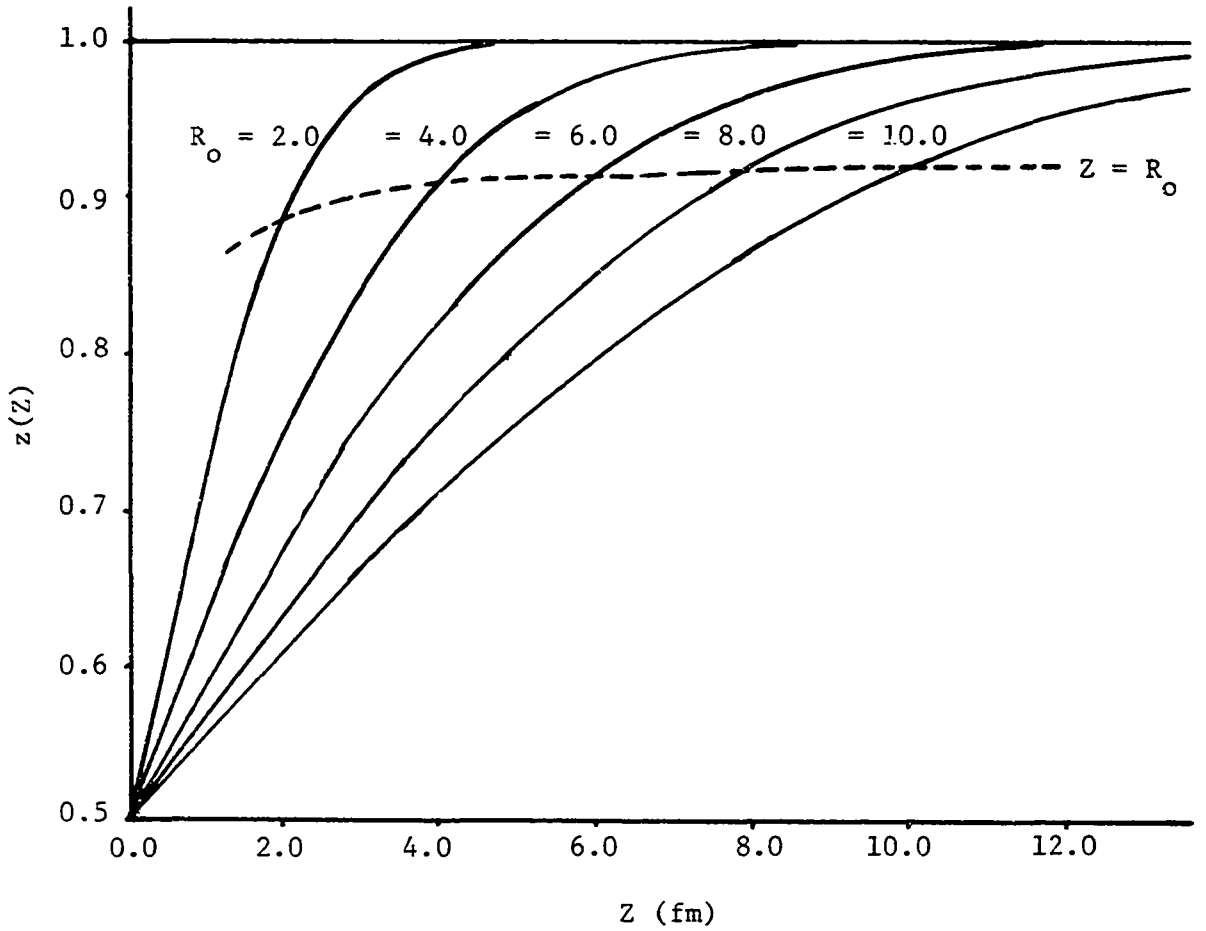


Figure 7. Behavior of the function z(Z) as the size parameter R_o increases and β is held fixed at 1.4 fm

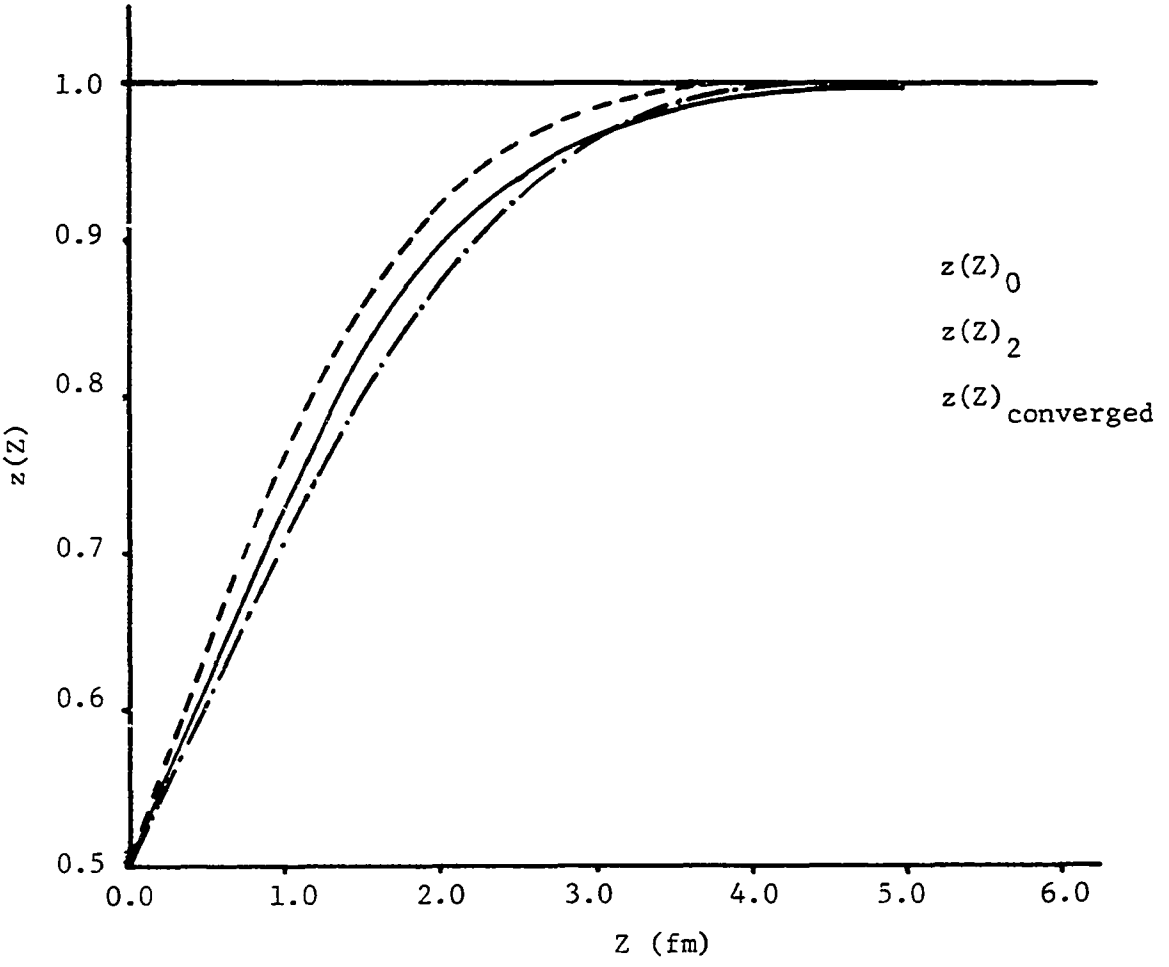


Figure 8. Comparison of the function z(Z) for zero-order, second-order and converged results

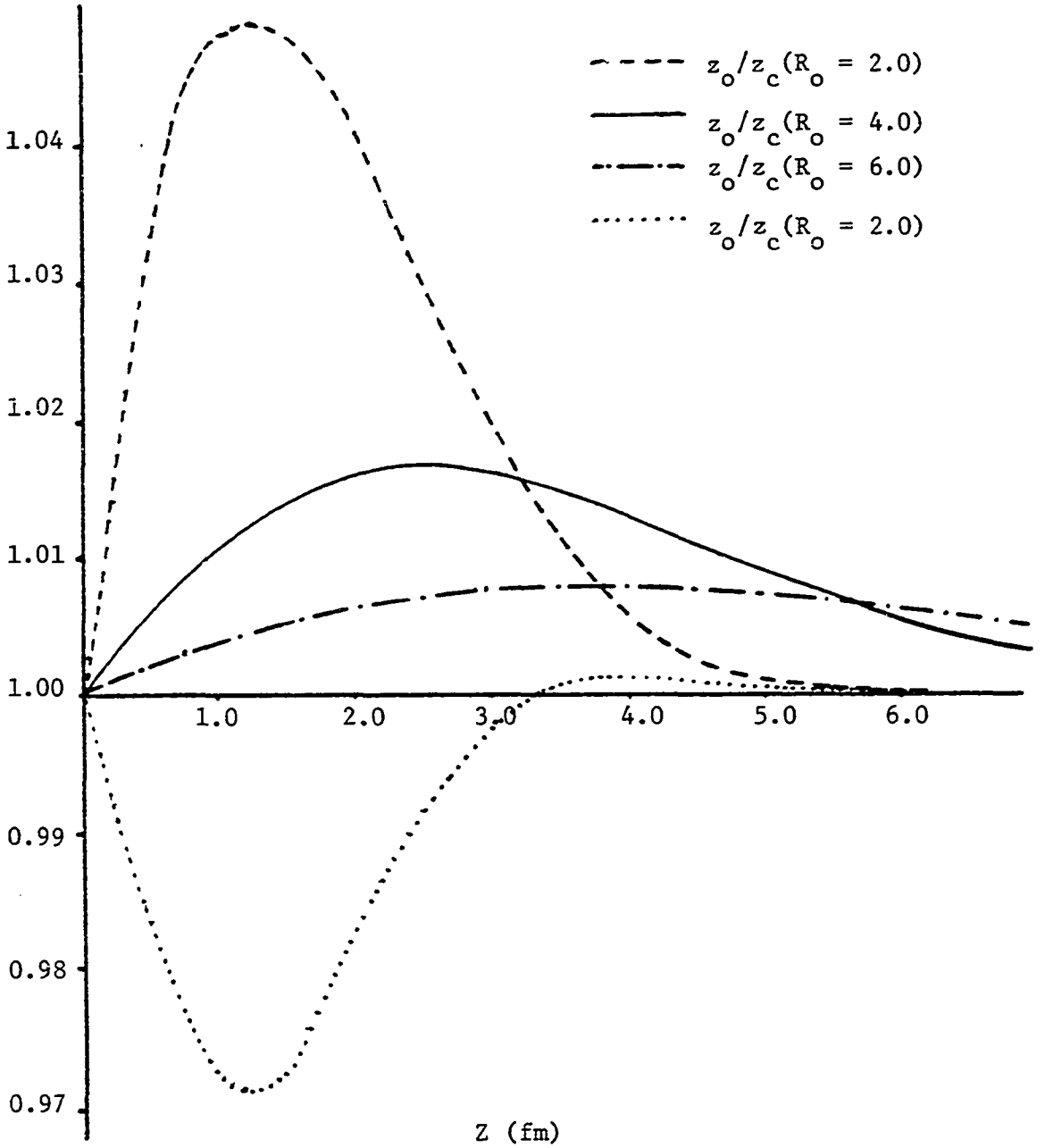


Figure 9. Comparison of ratio of $z_n/z_{converged}$ for R_o = 2.0, 4.0, 6.0 fm and β = 1.4 fm

$$w^{\pm}(\vec{r}) \simeq \Gamma(b)\ \theta^{\pm}(Z) \quad . \tag{4.22}$$

Except for the very smallest nuclear systems, the approximation Eq. (4.22) should be adequate. This conclusion is reinforced by noting that real nuclear densities may be expected to be considerably softer than are Gaussian densities. The equivalent R_o values would be larger and the convergence enhanced.

B. The Deuteron Vertex Function and Sensitivities

At low energies (less than ∿100 MeV) a common approach to the calculation of the transition amplitude Eq. (3.49) has been to replace the vertex functions $d_{LM}(\vec{r})$ by point vertex functions (51) of the form

$$d_{LM}(\vec{r}_P - \vec{r}_N) \approx d^o_{LM}\ \delta(\vec{r}_p - \vec{r}_n) \quad . \tag{4.23}$$

It has been standard practice as well to retain only the S-state vertex function under the assumption that this term dominates the (p,d) reaction. While this is certainly true at low enough energy ($\lesssim$40 MeV), it is worth examining the Fourier transform of the vertex functions to obtain some insight into the relative contributions of the S-state and D-state versus momentum mismatch $\vec{\Delta}$. The Fourier transform of the deuteron vertex function is defined by

$$\tilde{d}_{LM}(\vec{Q}) = \int e^{i\vec{Q}\cdot\vec{r}}\ d_{LM}(\vec{r})\ d^3r \tag{4.24a}$$

or

$$\tilde{d}_{LM}(\vec{Q}) = (i)^L\ Y_{LM}(\hat{Q})\ D_L(Q) \quad , \tag{4.24b}$$

where

$$D_L(Q) = 4\pi \int_0^\infty r\, j_L(Qr)\, U_L^{eff}(r)\, dr \quad . \tag{4.25}$$

The momentum mismatch $\vec{\Delta} = \vec{k}_p - \vec{k}_d/2$ is a function of the incident proton kinetic energy and the mass of the target. Figure 10 plots the vertex function $D_L(\Delta)$ versus Δ, and the incident kinetic energy against Δ for forward scattering. As can be seen at low momentum mismatch, the S-state is clearly dominant, whereas between 1.3 and 3.4 fm^{-1} the D-state appears to be dominant. At about 500 MeV proton kinetic energy the S-state contribution is virtually zero. The forward cross section for 700 MeV kinetic energy is dominated by the D-state, although there is a significant admixture of the S-state as well.

In order to address the questions about sensitivities in a systematic fashion I begin by rewriting Eq. (3.49) for the transition amplitude with the Fourier transforms of the vertex functions. Thus

$$T(L_A m_A;\, L_B m_B\, L_D m_D) = \left(\frac{1}{2\pi}\right)^6 \int d^3k\, \tilde{d}_{L_D m_D}(\vec{Q}_p - \vec{k}) \times \int d^3r_n d^3r_p\, e^{i\vec{Q}\cdot\vec{r}_n + i\vec{k}\cdot(\vec{r}_p - \vec{r}_n)}\, N^{L_B m_B}_{L_A m_A}(\vec{r}_p, \vec{r}_n) \quad , \tag{4.26}$$

and

$$\vec{Q} = \vec{Q}_p + \vec{Q}_n \quad . \tag{4.27}$$

The zero-range approximation may be obtained by examining the Taylor

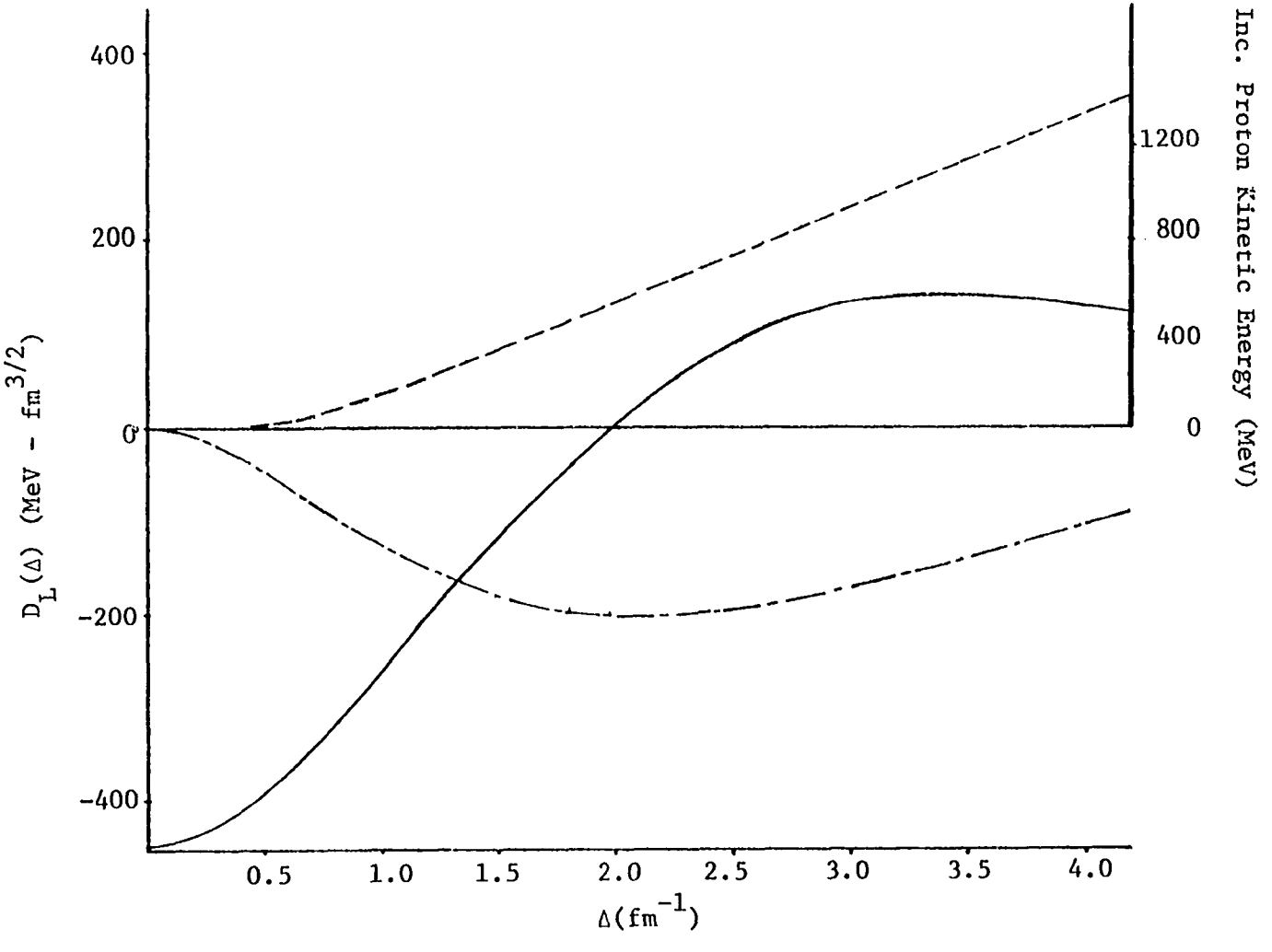


Figure 10. The solid curve is the S-state of $D_L(\Delta)$ and the dot-dash curve shows the behavior the D-state of $D_L(\Delta)$ plotted against the momentum transfer Δ; the dashed curve shows the momentum mismatch between the proton and deuteron for 0° scattering over the range of proton kinetic energy

series expansion of the vertex functions $\tilde{d}_{L_D m_D}(\vec{Q}_p - \vec{k})$ about $\vec{k} = 0$. This may be written as

$$\tilde{d}_{L_D m_D}(\vec{Q}_p - \vec{k}) = \sum_{n=0}^{\infty} \frac{(-\vec{k} \cdot \vec{\nabla}_k)^n}{n!} \tilde{d}_{L_D m_D}(\vec{k}) \Big|_{\vec{k}=\vec{Q}_p} . \tag{4.28}$$

Substitution of Eq. (4.28) into Eq. (4.26) yields for the transition amplitude the expression

$$T = \sum_{n=0}^{\infty} T_n \tag{4.29}$$

where the n-th term is defined as

$$T_n = \frac{1}{(2\pi)^6} \int d^3k \, d^3r_p \, d^3r_n \left[\frac{(-\vec{k} \cdot \vec{\nabla}_{Q_p})^n}{n!} \tilde{d}_{L_d m_d}(\vec{Q}_p) \right] \times e^{i\vec{Q}\cdot\vec{r}_n + i\vec{k}\cdot(\vec{r}_p - \vec{r}_n)} N^{L_B m_B}_{L_A m_A}(\vec{r}_p, \vec{r}_n) . \tag{4.30a}$$

The n = 0 zero term is

$$T_o = \frac{1}{(2\pi)^3} \tilde{d}_{L_d m_d}(\vec{Q}_p) \int d^3r \, e^{i\vec{Q}\cdot\vec{r}} N^{L_B m_B}_{L_A m_A}(\vec{r}, \vec{r}) . \tag{4.30b}$$

I shall refer to this term as the zero-range trunction (ZRT). The ZRT has the advantage over the zero-range approximation that the strength of the vertex constants are specified rather than adjusted from a phenomenological basis. Furthermore, it includes in a natural manner both the S-state and D-state strengths. The general terms for arbitrary

n may be simplified by repeated use of the divergence theorem and the chain rule for differentiation. Under these manipulations Eq. (4.30a) becomes

$$T_n = \frac{1}{(2\pi)^3} \int d^3r_n \, d^3r_p \, \delta(\vec{r}_p - \vec{r}_n) \, e^{i\vec{Q}\cdot\vec{r}_n}$$

$$\times \frac{(i\vec{\nabla}_{r_p} \cdot \vec{\nabla}_{Q_p})^n}{n!} N^{L_B m_B}_{L_A m_A}(\vec{r}_p, \vec{r}_n) \, \tilde{d}_{L_d m_d}(\vec{Q}_p) \quad . \tag{4.31}$$

A simple examination reveals that Eq. (4.31) reduces to Eq. (4.30b) for n = 0 as it should. The form of Eq. (4.31) makes it clear that unless the series Eq. (4.28) converges very rapidly in the energy regime of interest, the 'simplification' introduced by Eq. (4.31) would not present any advantage over the exact finite range calculation (52), since the gradient mixing terms may be expected to be difficult to evaluate simply.

In order to compare the ZRT with the exact finite range vertex functions, it is useful to construct the model calculations within an analytic framework. To this end the vertex functions $d_{L_d m_d}(\vec{r})$ have been fit to a sum of Gaussians (see Tables 1 and 2) as are the neutron wavefunctions (see Table 3).

In Section A the behavior of the nucleon-nucleon distortion factors $w^{\pm}(\vec{r})$ was discussed. We found that a reasonable truncation of the series representation may be

$$w^{\pm}(\vec{r}) \approx w^{\pm}_o(\vec{r}) = \Gamma^{\pm}(b) \, \theta^{\pm}(Z) \quad . \tag{4.32}$$

Table 1. Gaussian fit coefficients of deuteron D-state vertex function generated from Reid soft core potential

Form: $U_D^{eff}(r) \doteqdot r^3 \sum_{j=1}^{M} C_j^D \exp(-\alpha_j^D r^2)$

$M = 25$

j	C_j^D	α_j^D
1	.2024014E + 05	.6684000E + 01
2	-.1098078E + 05	.4856000E + 01
3	-.9282914E + 05	.2025500E + 02
4	-.5080554E + 01	.6063700E - 01
5	.2459807E + 05	.2668785E + 02
6	.7551059E + 05	.1846687E + 02
7	-.1506456E + 05	.8345700E + 01
8	.3837713E + 04	.3812730E + 01
9	-.7432679E + 03	.2346205E + 01
10	.4953998E + 01	.6438500E + 00
11	-.3612253E + 02	.8784360E + 00
12	-.2925754E + 01	.4166280E + 00
13	.7442443E + 00	.2770020E + 00
14	-.8358696E + 00	.2190330E + 00
15	.5911094E + 00	.1690240E + 00
16	-.4100525E + 00	.1398840E + 00
17	.3970916E + 01	.5520000E - 01
18	.2951982E + 00	.9918500E - 01
19	-.2152051E + 01	.7357200E - 01
20	.4522515E + 01	.6751000E - 01

Table 1. Continued

j	c_j^D	α_j^D
21	.1377630E - 01	.3280000E - 01
22	.1285684E + 00	.4050000E - 01
23	-.1239874E - 02	.2940000E - 01
24	-.1524743E + 01	.5124000E - 01
25	-.6306129E - 01	.3650000E - 01

Table 2. Gaussian fit coefficients of deuteron S-state vertex function generated from Reid soft core potential

Form: $U_S^{eff} \doteq r \sum_{j=1}^{M} C_j^S \exp(-\alpha_j^S r^2)$

$M = 25$

j	C_j^S	α_i^S
1	.6108439E + 04	.6684000E + 01
2	-.4936616E + 04	.4856000E + 01
3	-.5106693E + 03	.2025500E + 02
4	-.4642542E + 02	.6063700E - 01
5	-.8365475E + 02	.2668700E + 02
6	.3246480E + 03	.1846687E + 02
7	-.3528151E + 04	.8324570E + 01
8	.3311058E + 04	.3812730E + 01
9	-.6071844E + 03	.2346205E + 01
10	-.7553494E + 02	.8784360E + 00
11	.9676435E + 01	.6438500E + 00
12	-.7028647E + 01	.4166280E + 00
13	.3517861E + 01	.2770020E + 00
14	-.4477011E + 01	.2190330E + 00
15	.4016737E + 01	.1690240E + 00
16	-.3094156E + 01	.1398840E + 00
17	.3652883E + 02	.5520000E - 01
18	.2501570E + 01	.9918500E - 01
19	-.1930382E + 02	.7357200E - 01
20	.4093158E + 02	.6751000E - 01

Table 2. Continued

j	C_j^S	α_i^S
21	.1299794E + 00	.3280000E - 01
22	.1201281E + 01	.4050000E - 01
23	-.1179464E - 01	.2940000E - 01
24	-.1408953E + 02	.5124000E - 01
25	-.5919600E + 00	.3650000E - 01

Table 3. Gaussian fit coefficients of a 1p 3/2 neutron wavefunction generated from density dependent Hartree-Fock code for carbon 12

Form: $U(r) = r^2 \sum_{j=1}^{M} C_j^n \exp(-\alpha_j^n r^2)$

$M = 13$

j	C_j^n	α_i^n
1	.2534095E + 00	.2100000E + 01
2	-.1925597E + 01	.1040000E + 01
3	-.3627780E - 01	.4300000E + 01
4	-.2562850E + 02	.5400000E + 00
5	-.3563548E + 01	.3450000E + 00
6	.7491653E + 00	.2340000E + 00
7	.1702169E + 02	.4550000E + 00
8	.1341978E + 02	.6560000E + 00
9	.1135376E + 00	.7820000E - 01
10	-.1948351E + 00	.6607000E - 01
11	.1944234E + 00	.5700000E - 01
12	-.1288989E + 00	.5000000E - 01
13	.4757558E - 01	.4650000E - 01

From the properties of the theta functions $\theta^{\pm}$ Eqs. (3.70a) and (3.70b) we see that

$$\theta^{+}(Z) + \theta^{-}(Z) = 1 \quad . \tag{4.33}$$

Since the functions Γ^{+} and Γ^{-} are both smooth functions having similar strength and range parameters, it is reasonable to replace the theta functions by their average values of 1/2. Thus the functions $\bar{w}_j$, Eq. (3.56), becomes

$$\begin{aligned}\bar{w}_j(\vec{r}_j,\vec{r}_p,\vec{r}_n) \doteq [&-\tfrac{1}{2}\Gamma^{+}(\vec{b}_p-\vec{b}_j) - \tfrac{1}{2}\Gamma^{-}(\vec{b}_p-\vec{b}_j) - \tfrac{1}{2}\Gamma^{-}(\vec{b}_n-\vec{b}_j) \\ &+ (\tfrac{1}{2}\Gamma^{+}(\vec{b}_p-\vec{b}_j) + \tfrac{1}{2}\Gamma^{-}(\vec{b}_p-\vec{b}_j)(\tfrac{1}{2}\Gamma^{-}(\vec{b}_n-\vec{b}_j))]\end{aligned} \tag{4.34}$$

where

$$\Gamma^{\pm}(b) = \frac{1}{2\pi\beta^2}\sigma_{NN}^{TOT(\pm)}(1 - i\alpha^{NN(\pm)})\exp(-b^2/\beta^2) \quad . \tag{4.35}$$

In principle the range β would also very from the incident to the exit channel. However the work of Vary and Dover (42) indicates that the range parameter is comparatively insensitive to the energy fluctuations. For the tests I make, β shall be fixed at 1.24 Fermi.

The total distortion function W of Eq. (3.55) may be written as a series in the number of nucleons in the residual target which participate in the scattering.

$$W = 1 - \sum_{j\varepsilon B}\bar{w}_j + \sum_{j\neq k}^{B}\bar{w}_j\bar{w}_k - \sum_{j\neq k\neq \ell}\bar{w}_j\bar{w}_k\bar{w}_\ell + \cdots \quad . \tag{4.36}$$

Recall that the function W appears in the integrated distortion factor

$F^{L_B m_B}_{\lambda_B \nu_B}(\vec{r}_p, \vec{r}_n)$, Eq. (3.59). Consider only diagonal scattering $(L_B m_B) = (\lambda_B \nu_B)$. Then F simplifies to

$$F(r_p, r_n) = \int \rho^{(B)}(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_B)\, W(\vec{r}_p, \vec{r}_n, \vec{r}_1, \cdots \vec{r}_B) d^3 r_1 \cdots d^3 r_B \tag{4.37}$$

Substitution of Eq. (4.36) into Eq. (4.37) then yields a spectator expansion in the residual target density.

$$F(\vec{r}_p, \vec{r}_n) = 1 + F_1(\vec{r}_p, \vec{r}_n) + F_2(\vec{r}_p, \vec{r}_n) + \cdots F_3(\vec{r}_p, \vec{r}_n) \tag{4.38}$$

$$F_n(\vec{r}_p, \vec{r}_n) = (-)^n \int \rho^{(n)}\, \bar{w}_1\, \bar{w}_2 \cdots \bar{w}_n\, d^3 r_1 \cdots d^3 r_n \tag{4.39}$$

where $\rho^{(n)}$ is the n-body density of the resident target. It is this density expansion which introduces nucleon-nucleon correlations. The scattering of the incident proton or exiting deuteron from target nucleon to target nucleon is affected by the strong short range correlations existing between the 'spectator' of the residual target.

In order to evaluate the finite range transition amplitude, it is necessary to specify the model in which this is to be done. To this end we shall consider the reaction cross sections obtained by retaining only the first two terms of the density expansion Eq. (4.38). If the target density is expressible as a Gaussian or a sum of Gaussians, the distortion F becomes an expression in Gaussians. Then the transition amplitude takes the form of the double folded asymmetric Gaussian which is discussed in Appendix D. The evaluation of the transition amplitude

is now expressible in terms of analytic functions. Suppose we consider the sensitivity of the cross section in this model to the details of the density. Table 4 contains the coefficients for the residual target density of carbon-11 fit to a sum of Gaussians. The density of the carbon-12 target was generated by a density dependent Hartree-Fock code (53) and the density of residual target was obtained by subtraction of the single particle density of the captured neutron. The values used for the total nucleon-nucleon cross section (σ_{NN}^{TOT}) and alpha (α^{NN}) are given in Table 5. Figure 11 displays the results of using the realistic density (Table 4) as opposed to single Gaussian densities (see Section A) with R_o = 1.826, 2.085, 2.355 fm. Notice that for R_o = 2.085 fm the cross section is virtually identical to that of the realistic density. This leads us to speculate that the radius of the single Gaussian density could be adjusted to yield a close resemblance to the realistic density, leading to a considerable simplification of the higher order terms in Eq. (4.38)

From Fig. 10 we obtained some indication of the relative importance of S and D-state contributions to the reaction cross section at different energies. In Figs. 12 through 16 the total differential cross section, the S-state and D-state contributions are displayed for the reaction $^{12}C(p,d)^{11}C(3/2^-g.s.)$ for a range of energies. We can see that the results are consistent with the systematics to be expected from Fig. 10.

How sensitive are the results to the zero-range truncation (ZRT)? In Figs. 17 through 22 the reaction cross section calculated through first order in the realistic density is displayed for both the FR and

Table 4. Gaussian fit coefficients for realistic Hartree-Fock carbon-11 density, normalized to unity

Form: $\rho(r) \doteq \sum_{j=1}^{M} C_j (\alpha_j/\pi)^{3/2} \exp(-\alpha_j r^2)$

$$\sum_{j=1}^{M} C_j = 1$$

$$M = 10$$

j	C_j	α_j
1	.411334549E - 03	.741894153E - 01
2	.845453470E - 01	.148378831E + 00
3	-.546794586E + 00	.222568246E + 00
4	.663778679E + 01	.296757661E + 00
5	-.264960114E + 02	.370947076E + 00
6	.604007273E + 02	.445136492E + 00
7	-.641846813E + 02	.519325907E + 00
8	.241154588E + 02	.593515322E + 00
9	.583670412E + 01	.667704738E + 00
10	-.484814650E + 01	.741894153E + 00

Table 5. The nucleon-nucleon cross sections σ_{NN}^{TOT} and alpha α^{NN} values[a] used for energy dependent comparison of finite range vertex functions and the zero-range truncation

Incident Kinetic Energy (MEV)	$\sigma_{NN}^{TOT}(+)$ (FM FM)	$\sigma_{NN}^{TOT}(-)$ (FM FM)	$\alpha^{NN}(+)$	$\alpha^{NN}(-)$
50	5.00	5.00	1.87	1.87
100	5.00	5.00	1.87	1.87
300	3.50	4.00	0.60	1.00
500	3.46	4.30	0.55	0.800
700	4.23	3.00	0.20	0.600
800	4.32	3.50	0.144	0.600
1050	4.39	3.46	-0.073	0.550

[a]The values listed in this table correspond to the values used in tests of finite range vertex function versus zero-range truncation computed through first order in density. They are meant to represent reasonable average values of these quantities at these energies. Some variation from accepted values may be expected.

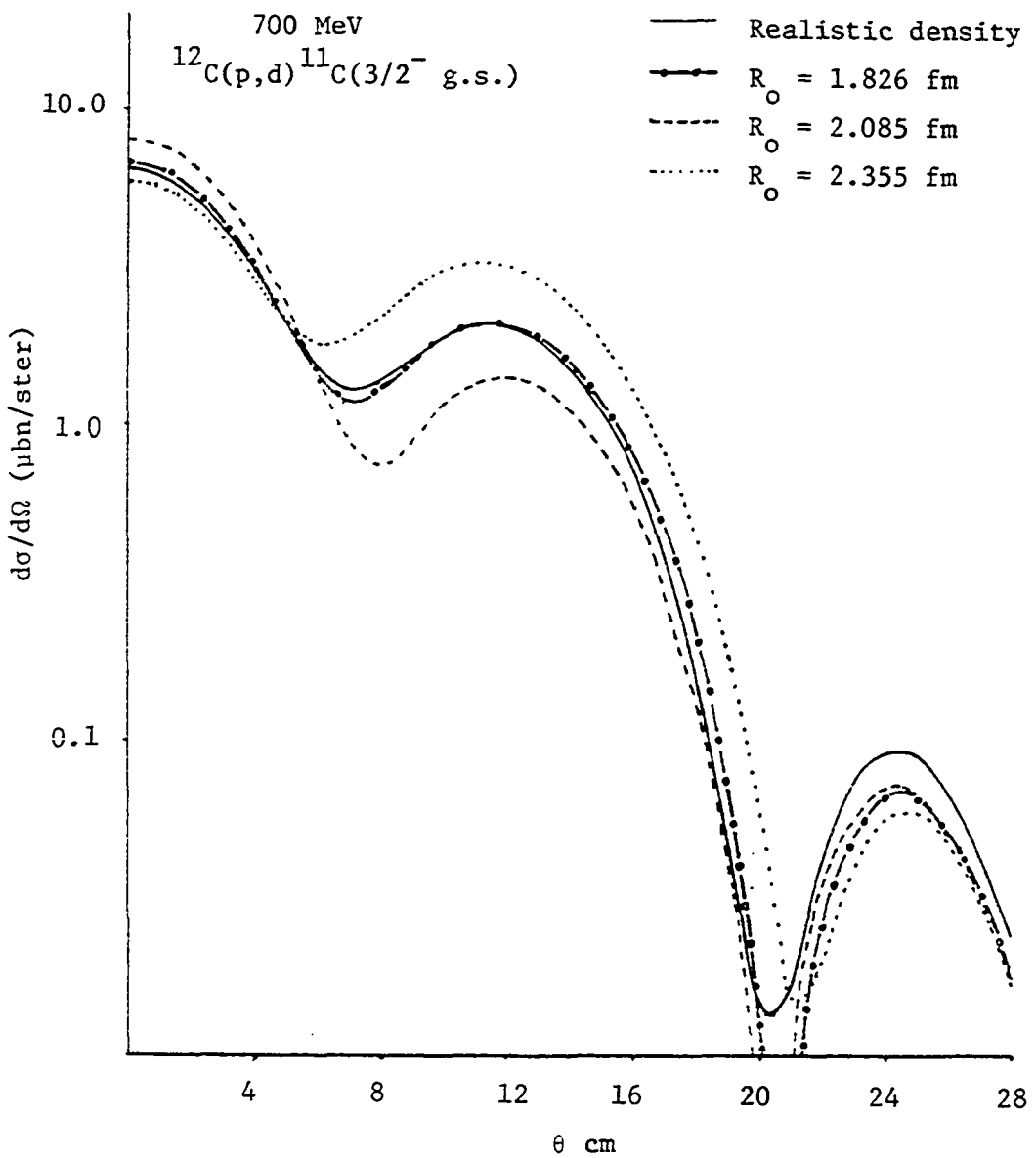


Figure 11. Sensitivity of first-order finite range calculation to realistic density and single Gaussian densities parameterized by R_o

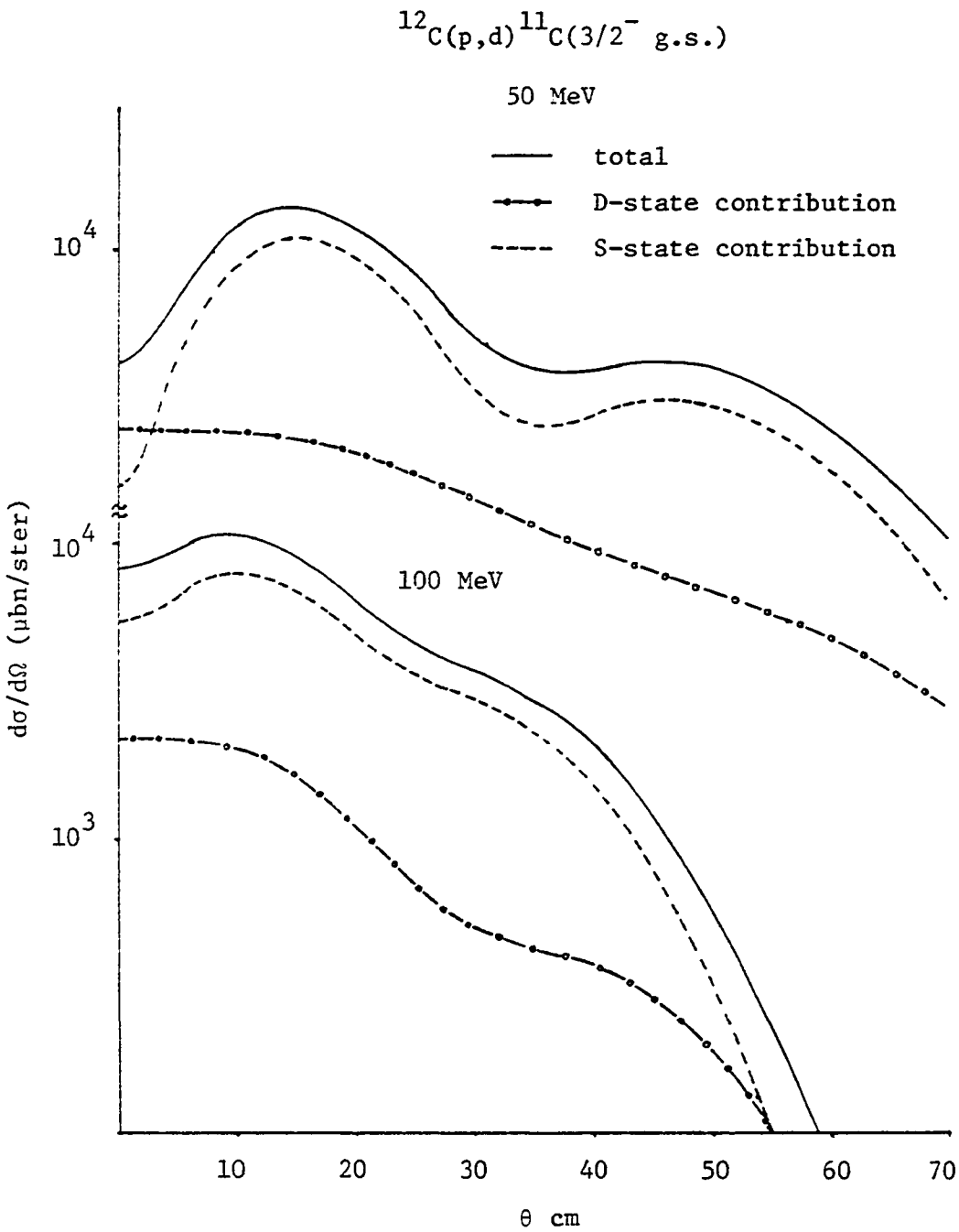


Figure 12. Comparison of deuteron S-state and D-state contributions to the differential cross section at 50 MeV and 100 MeV incident proton kinetic energy

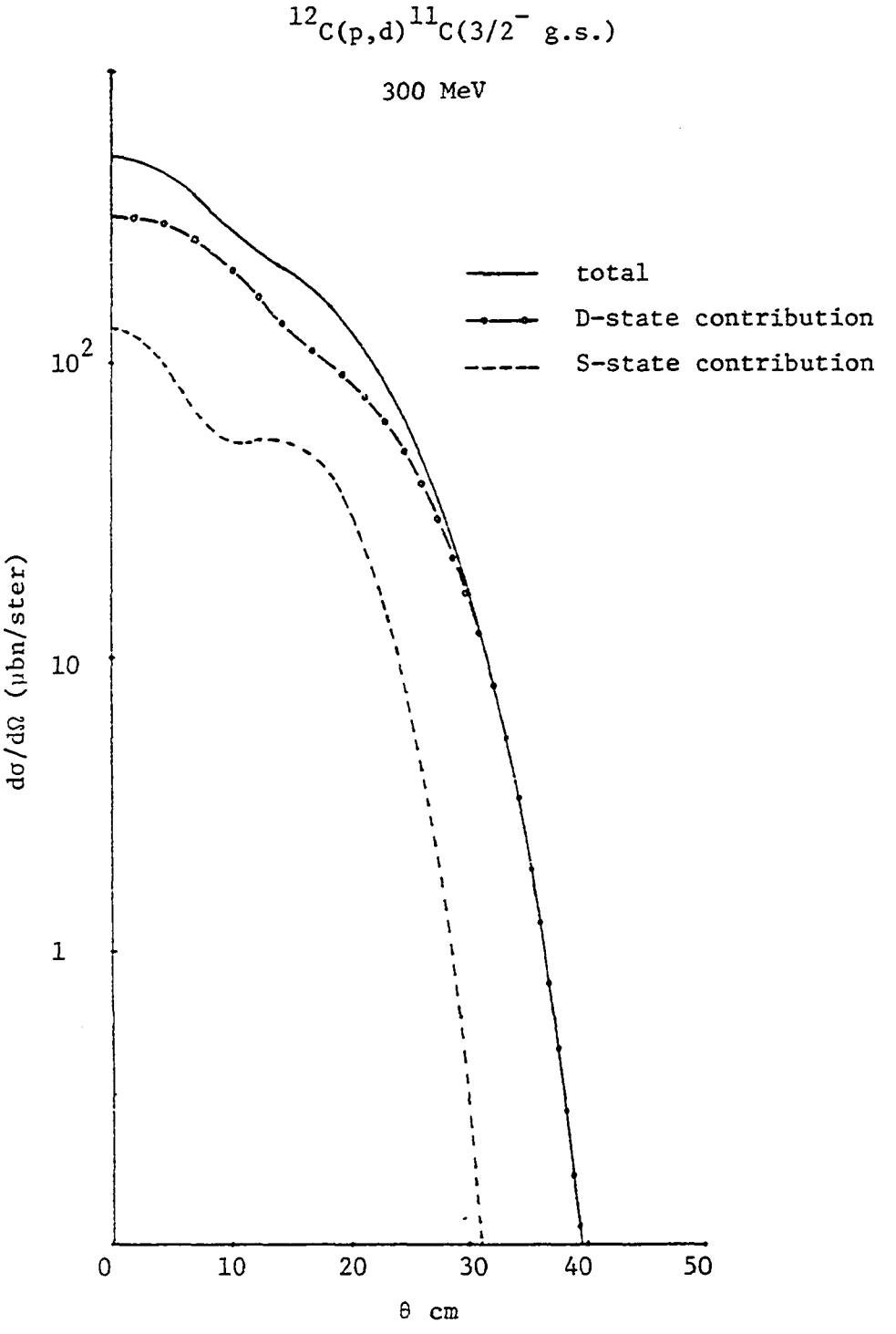


Figure 13. Comparison of deuteron S-state and D-state contributions to differential cross section at 300 MeV proton kinetic energy

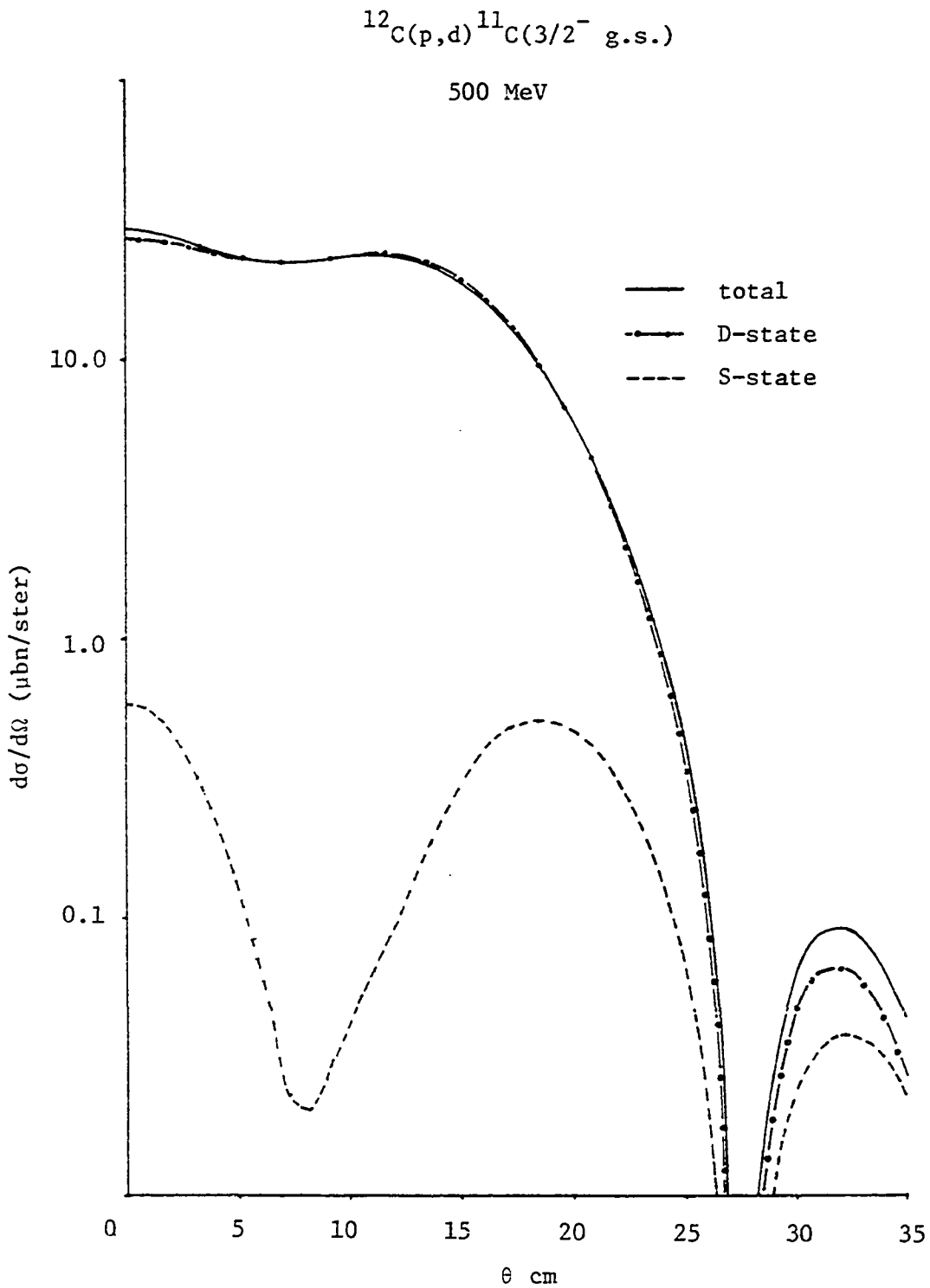


Figure 14. Comparison of deuteron S-state and D-state contributions to differential cross section of 500 MeV proton kinetic energy

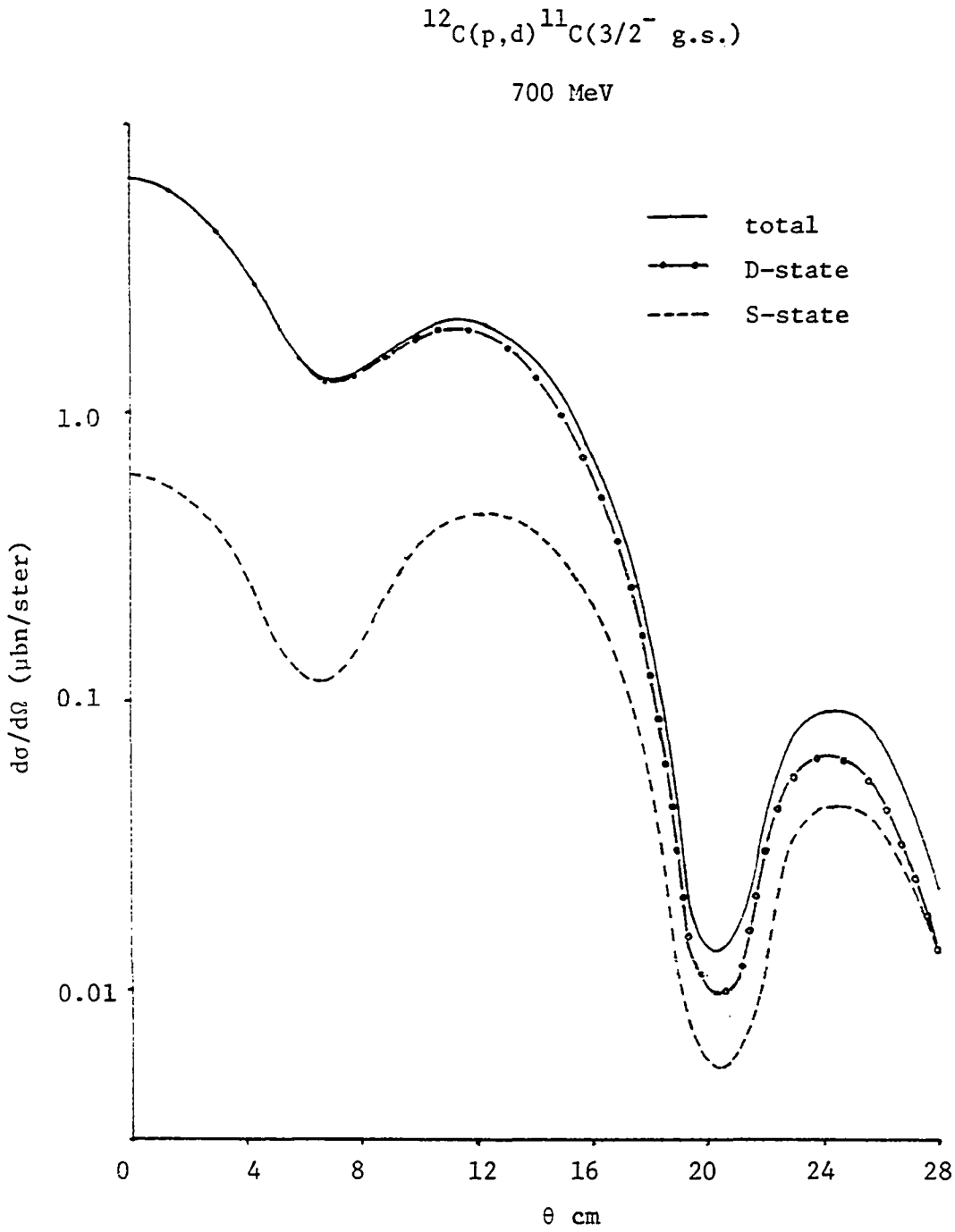


Figure 15. Comparison of deuteron S-state and D-state contributions to differential cross section at 700 MeV proton kinetic energy

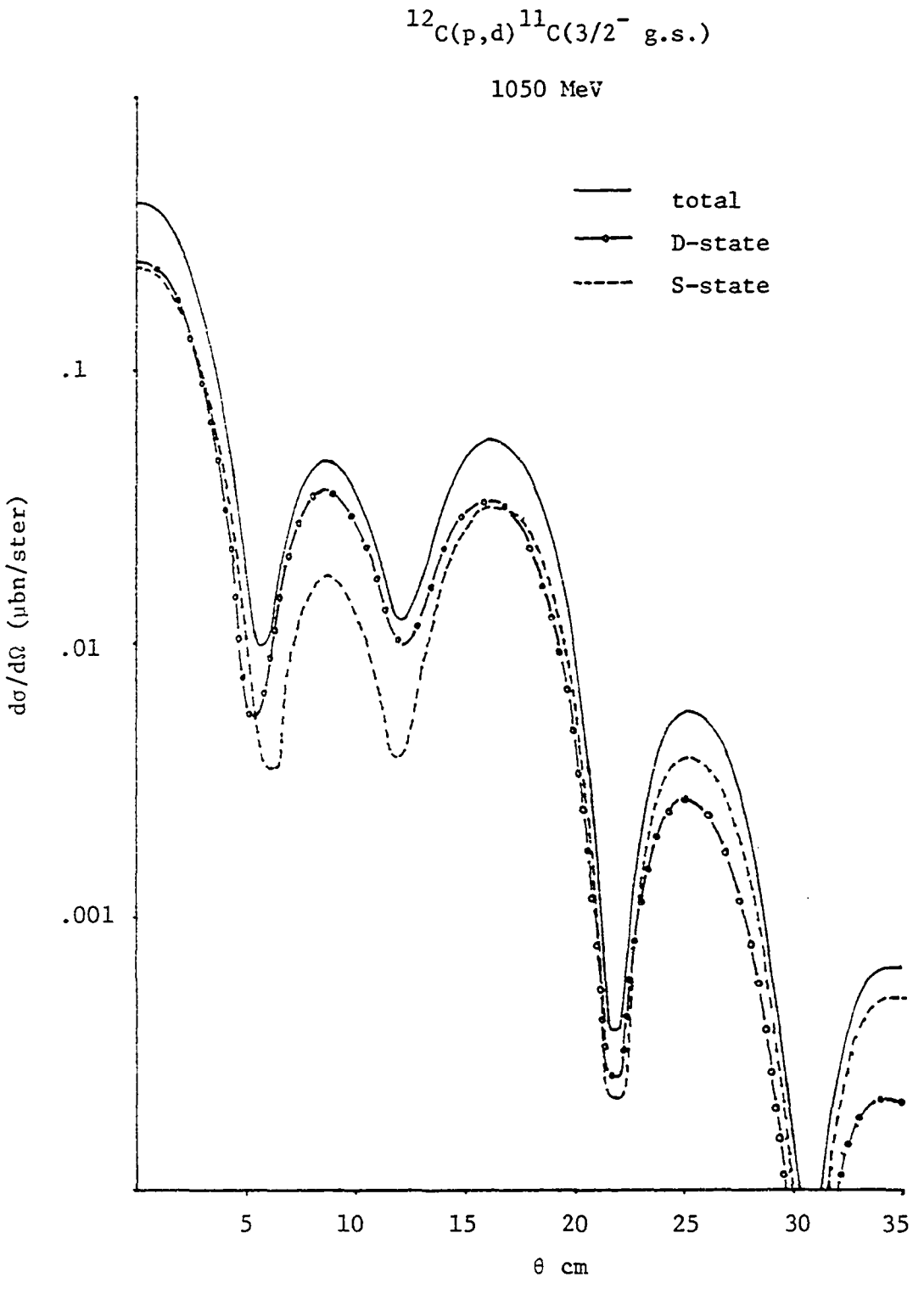


Figure 16. Comparison of deuteron S-state and D-state contributions to differential cross section at 1050 MeV proton kinetic energy

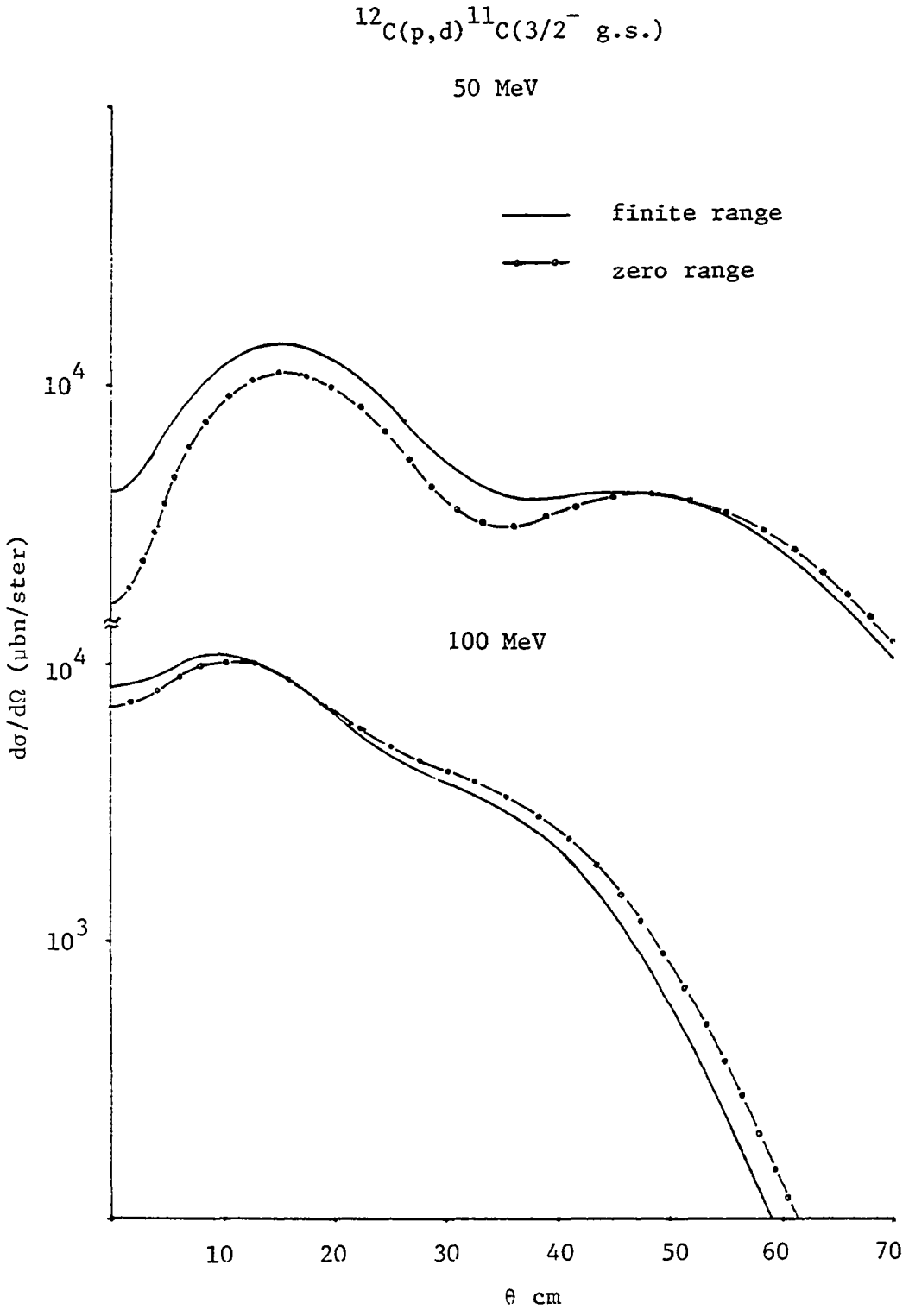


Figure 17. Comparison of finite range versus zero range deuteron vertex function at 50 MeV and 100 MeV

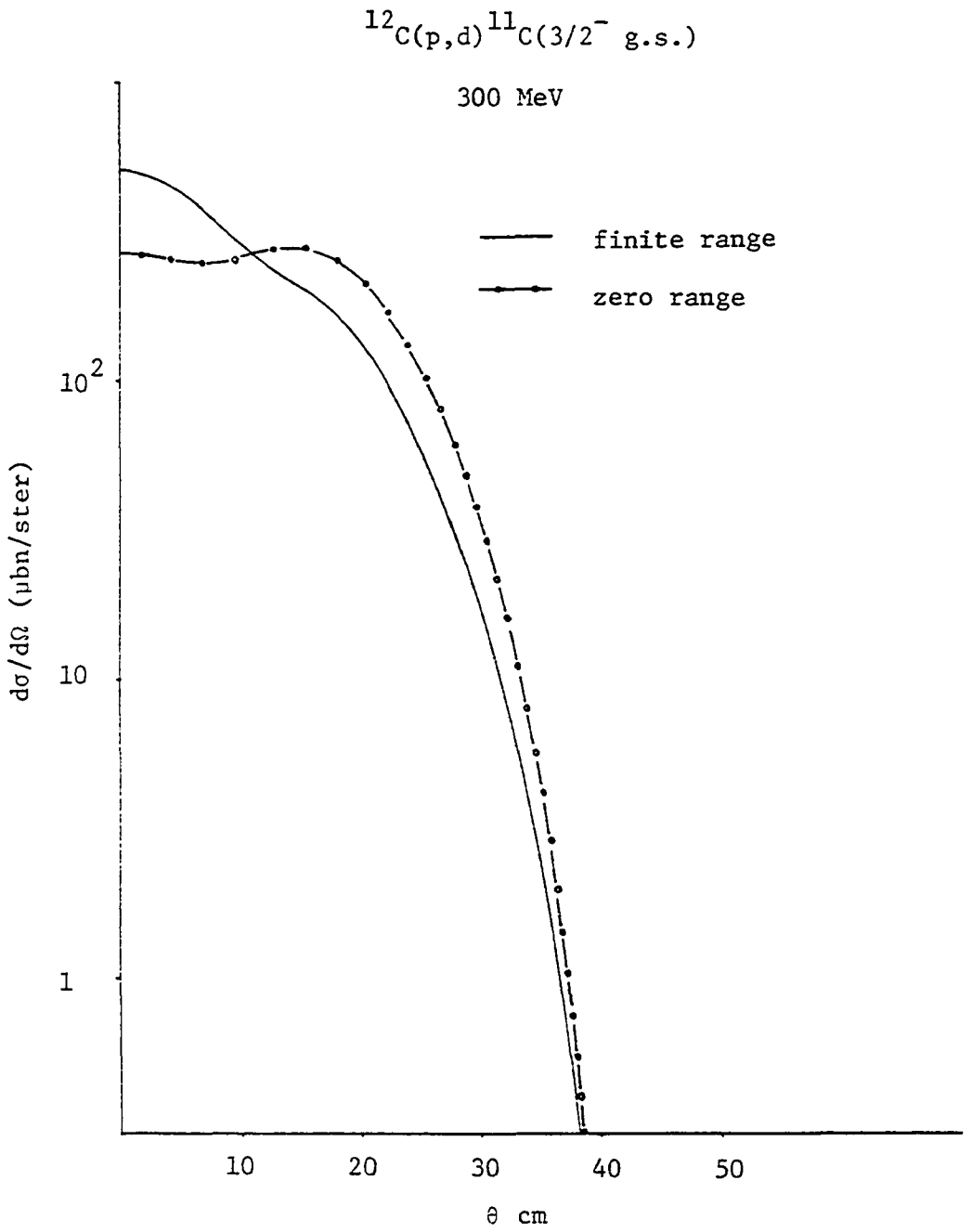


Figure 18. Comparison of finite range versus zero range deuteron vertex function at 300 MeV

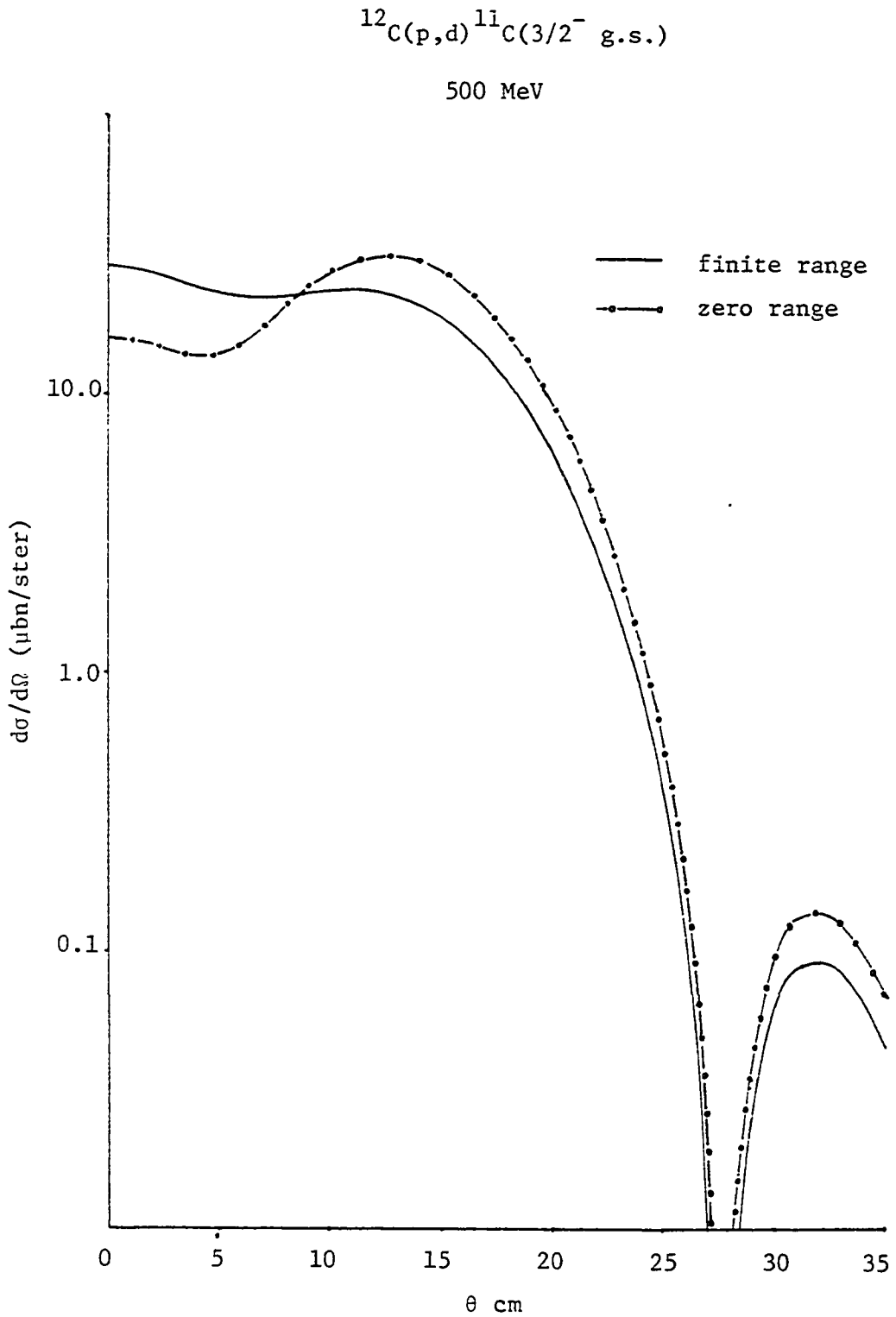


Figure 19. Comparison of finite range versus zero range deuteron vertex function at 500 MeV

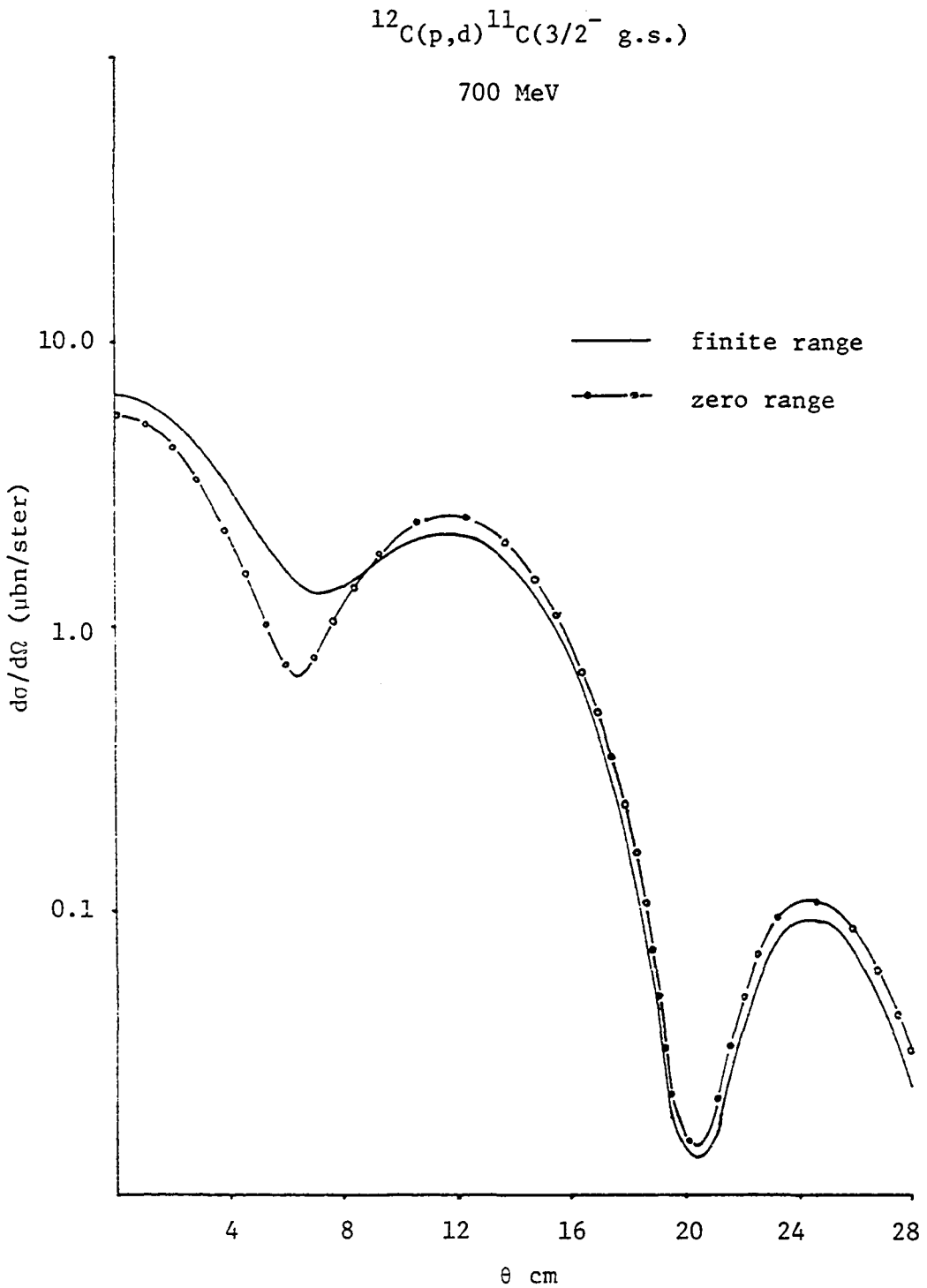


Figure 20. Comparison of finite range versus zero range deuteron vertex function at 700 MeV

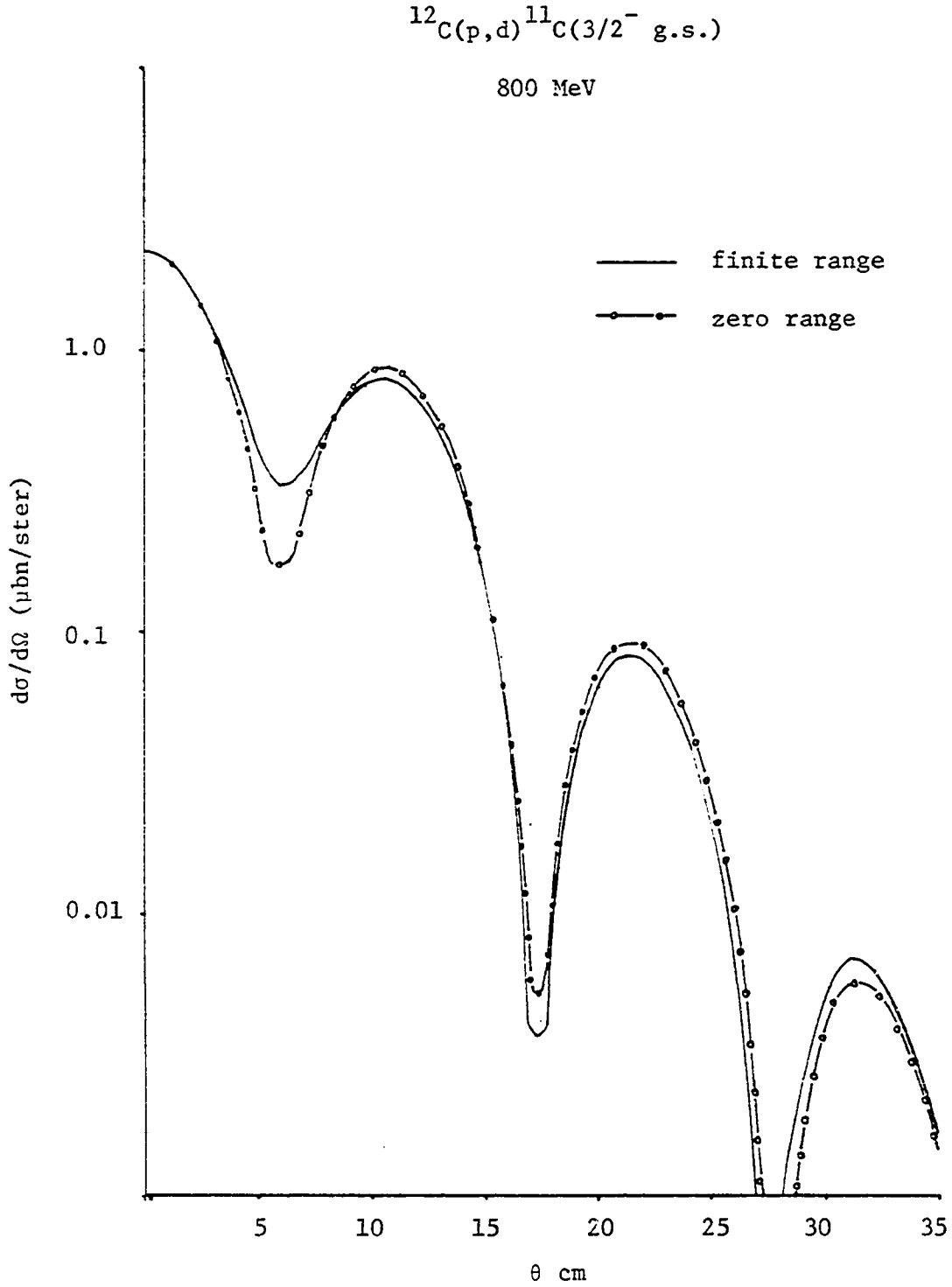


Figure 21. Comparison of finite range versus zero range deuteron vertex function at 800 MeV

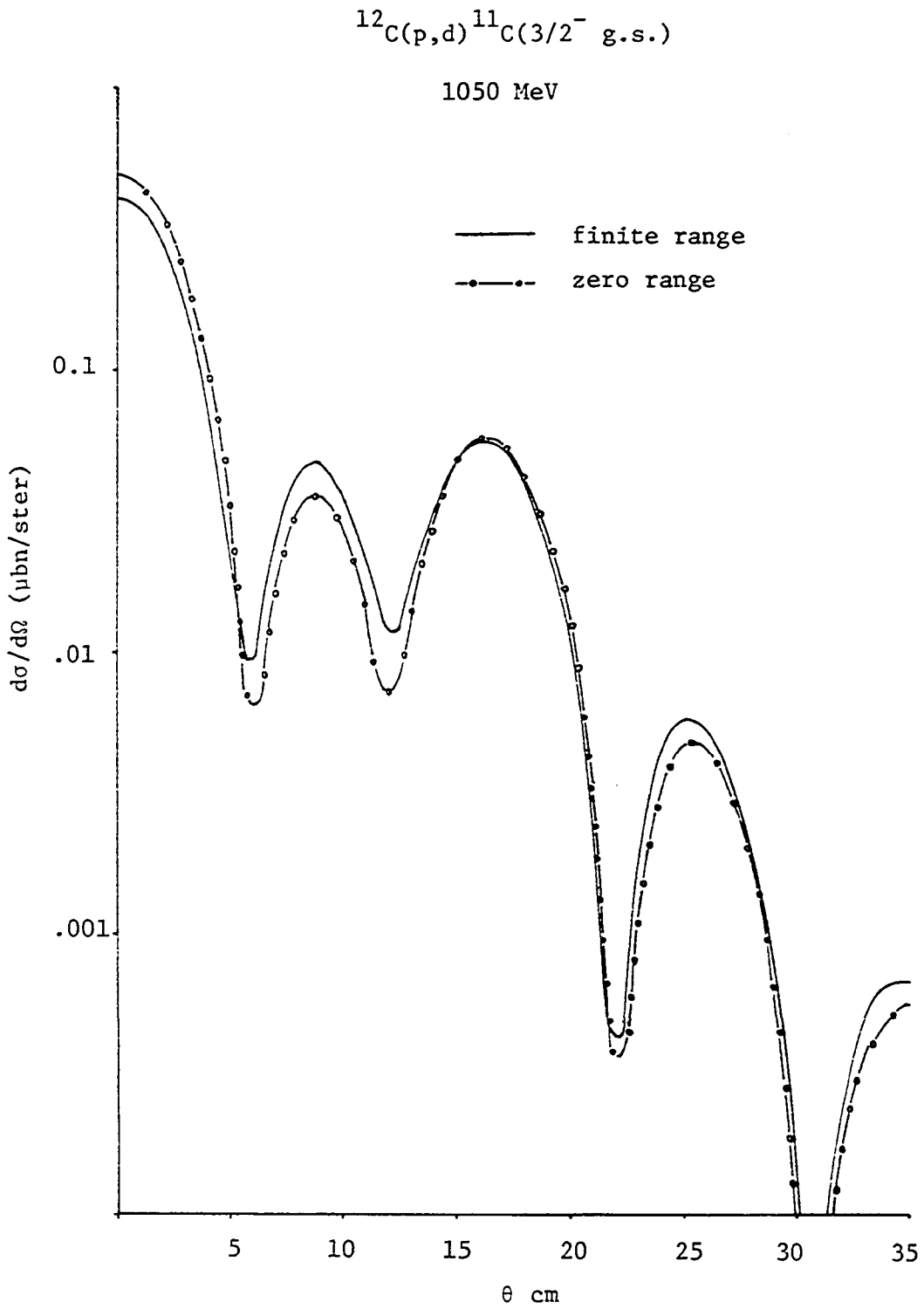


Figure 22. Comparison of finite range versus zero range deuteron vertex function at 1050 MeV

ZRT. It is noteworthy that throughout the energy range displayed, the ZRT appears to conform reasonably with FR calculation; for low energies the primary difference appears to be a slight rotation of the cross section. As the energy increases the differences become more pronounced, although the effects may be attributable to enhancement of a pointlike characterization of the deuteron. The diffraction minima at higher energies appear to sharpen. For order of magnitude effects and qualitative features, the ZRT reproduces the FR calculations throughout the energy range examined. Detailed agreement is however not present. It is conceivable that the first order correction to ZRT may be adequate to reduce the differences between the use of truncated representation of the vertex function and the exact finite range for computations. However, the magnitude of the effort to include this first order correction will, in some cases, be comparable to the effort to treat the full finite range.

Thus far we have examined certain of the qualitative sensitivities of the cross section through a first order in the density calculation. Suppose the many-body density if factorable, that is

$$\rho^{(n)}(1,2,\cdots n) = \prod_{i=1}^{(n)} \rho^{(1)}(i) \quad . \tag{4.40}$$

Then function F becomes

$$F(\vec{r}_p,\vec{r}_n) = [1 - \int \rho(r)\ \bar{w}(\vec{r},\vec{r}_p,\vec{r}_n)d^3r]^B \quad . \tag{4.41}$$

In the ZRT this immediately simplifies the details of the computation

of the transition amplitude. Figure 23 shows the cross section for zeroth, first, second, and total distortion calculations. As can be seen, the first order calculation dominates the forward cross section. The iterated calculation displays the expected diffractive characteristics.

In general, the many-body density is not factorable into a simple product of one-body densities. However, a parameterization which is sometimes used to account for the short range nucleon-nucleon correlations in the density (54,55) is written in the form

$$\rho^A(1,2,3, \cdots A) = (\prod_i^A \rho_i)(\prod_{j<k}^A C_{jk}) \quad . \tag{4.42}$$

where the ρ_i are the single particle densities and C_{jk} are the nucleon-nucleon correlation functions. I shall parametrize the C_{jk} as

$$C_{jk} = N(1 - \lambda \exp(-\beta_c r_{jk}^2)) \quad . \tag{4.43}$$

Here N is a normalization factor, λ is the correlation strength and β_c is a measure of the correlation range. Figure 24 compares a second order in the density ZRT calculation for $\lambda = 0$, $\beta_c = 0$ and $\lambda = 1$, $\beta_c = 0.60$ fm^{-2}. As can be seen there is a perceptible difference.

Finally we test the sensitivity to the density in a second order ZRT calculation. Figure 25 shows the cross section for the realistic density and the single Gaussian density $R_o = 2.085$ fm which in Fig. 10 was seen to correspond very closely with FR result with a realistic density. There is a remarkably close agreement between the results of

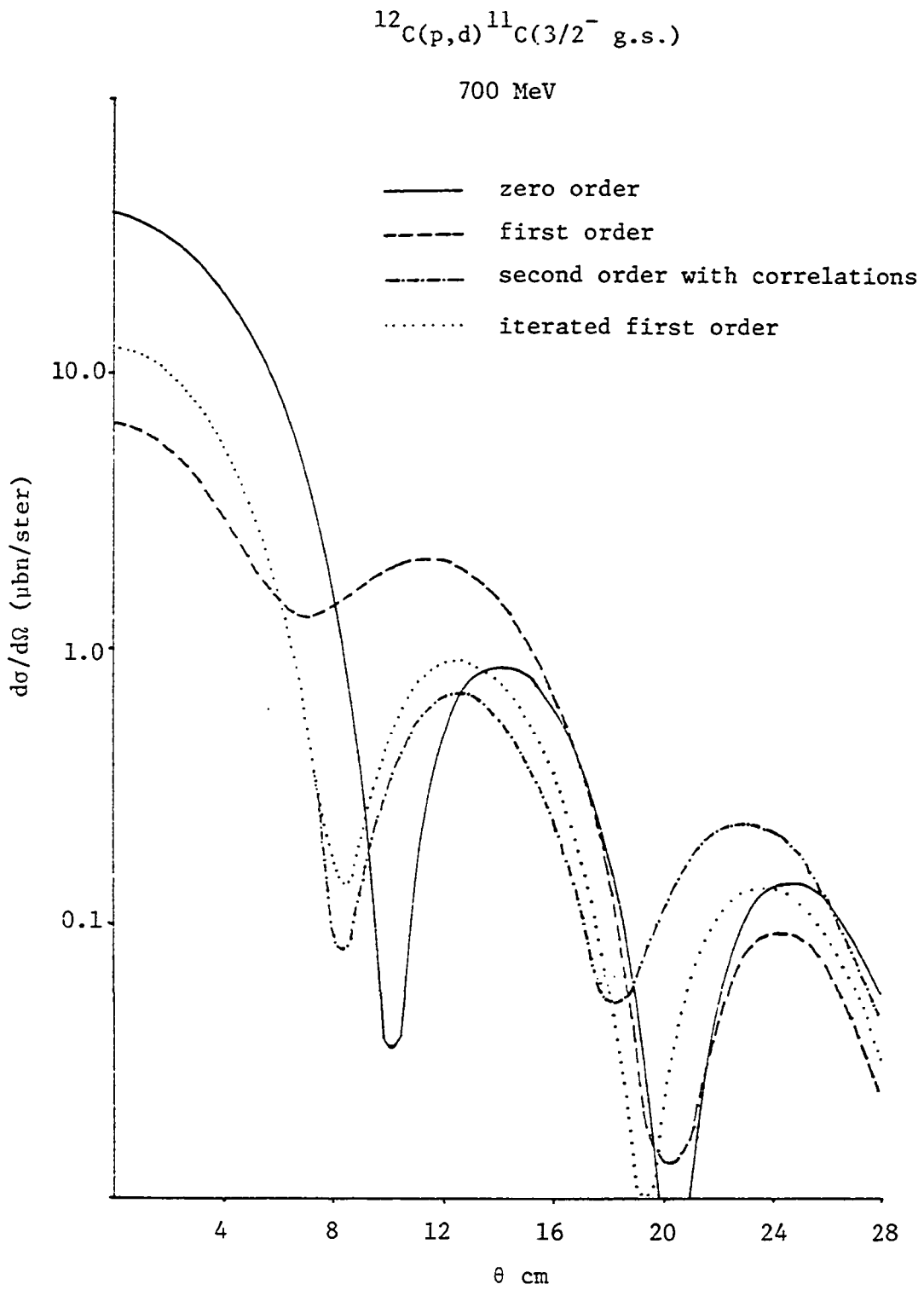


Figure 23. Comparison in zero-range calculation of differential cross section for zero order, first order, second order with correlations in density, and iterated first order without correlations

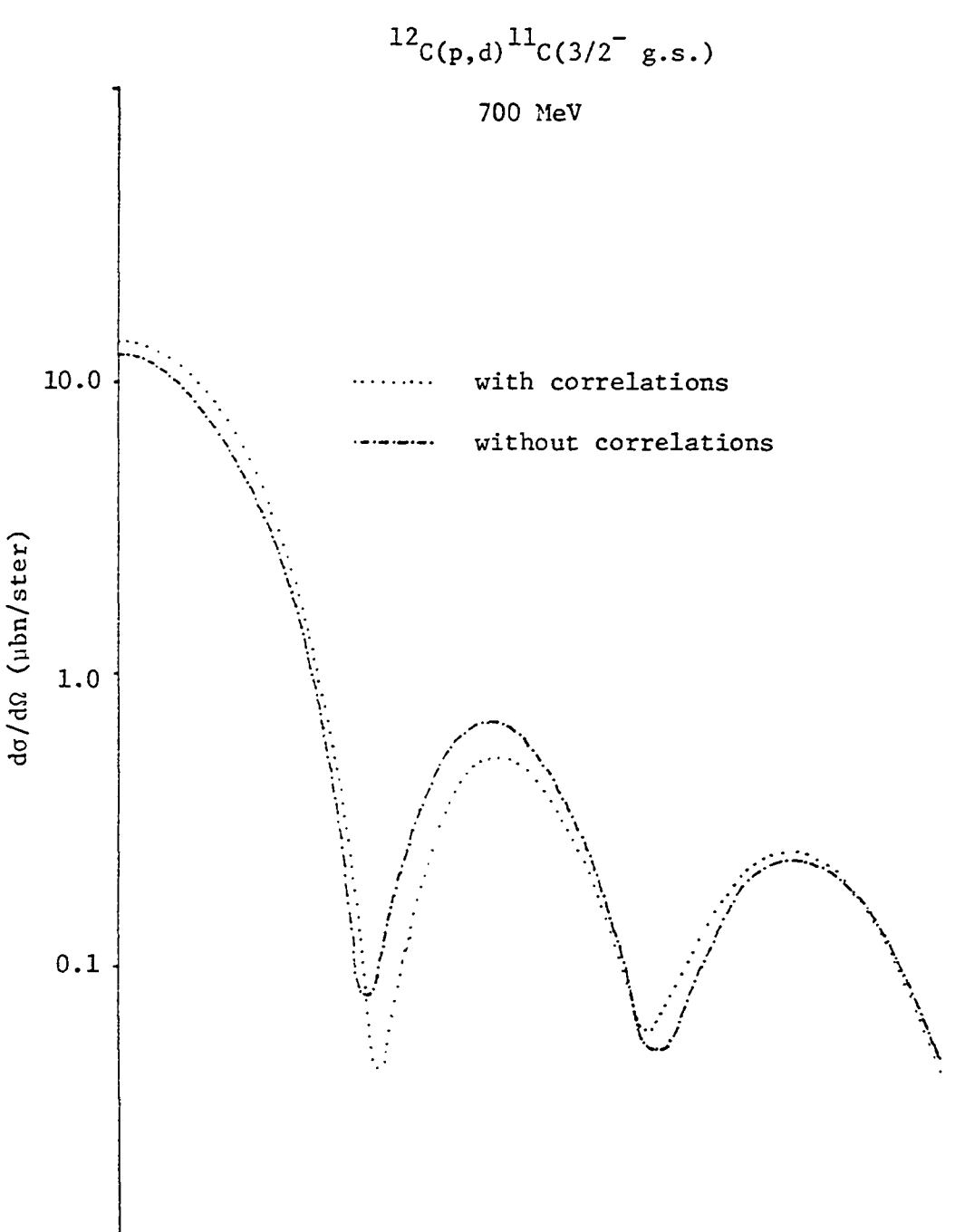


Figure 24. Comparison of zero range calculation through second order in realistic density with correlations and without correlations

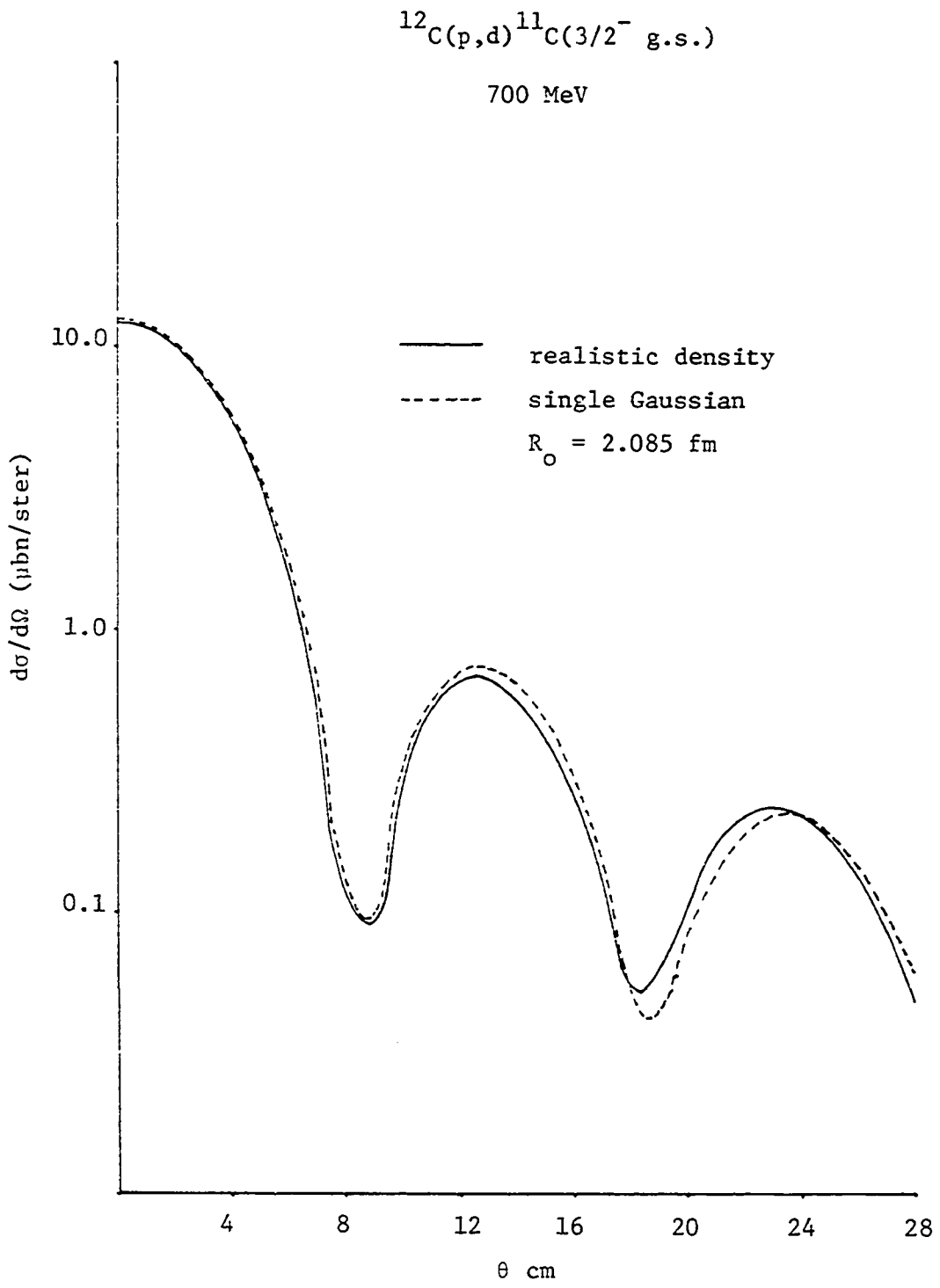


Figure 25. Comparison of zero range calculation through second order in density with correlations for realistic Hartree-Fock density and a single Gaussian density parameterized with R_o = 2.085 fm

the single Gaussian density with those of the realistic density. This leads to the conjecture that there is a critical size associated with the density from which the scattering is dominated.

C. Conclusions and Conjectures

Beginning with a Hamiltonian framework I have described the transition operator T for the interaction of two composite systems in which arbitrary multi-body potentials are allowed to exist. I have demonstrated how few-body mean fields may be ascribed to the averaging over the many-body interactions, or alternatively, how auxiliary potentials can be introduced. In Appendix A an algebraic identity and a functional theorem have been presented which facilitate the reduction of N-body operators to finite sums of arbitrary fewer body operators. The application of these mathematical identities to cluster-cluster interactions has yielded a generalized correlation expansion, referred to as the generalized spectator expansion (GSE). The GSE reduces in the appropriate limits to the correlation expansion of ELMT (23) or to the spectator expansion of ST (24). It contains as well the content of the Watson multiple scattering (14) and that of KMT (16). Furthermore this framework encompasses the subcluster-subcluster correlations in a physically transparent style. The flexibility inherent in these operators has been delineated, and a method for systematically decoupling a many-body operator into more manageable subpieces has been described.

The flexibility of the GSE makes the connection between the exact definition of a many-body operator and approximate treatments, such as

impulse approximation, fixed scatterer, closure, etc., evident. It places them in the proper perspective as leading order terms of exact but different decompositions of the operators. Physically motivated arguments have been used to select apposite definitions of propagators and other operators. Emphasis has been placed upon the connection of such arbitrary quantities with the Hamiltonian. The nature of the GSE casts a more fundamental aspect upon reactions which have hitherto been addressed primarily by phenomenology. Note especially, the correlated cluster treatment of high energy back-angle proton scattering by Fujita and Hüfner (40). Such collective effects as alpha-clustering in the scattering of even-even nuclei may also be addressed by this framework.

In illustration of the adduced flexibility for elastic and inelastic scattering, the reaction operator k has been defined. This makes provision for mean field effects and Pauli restrictions on intermediate state scattering (of fermions). In the two-body pieces, the methodology resembles that of the Brueckner reaction matrix of nuclear structure calculations. Also specified has been the cluster-cluster optical potentials through the use of projection operators. Finally the formation of matrix elements for two-body operators was sketched and the embedding of these elements in the local one-body densities of the interacting clusters demonstrated. This provides a basis for the commonly used double folded optical potential.

While the spectator expansion is not directly applicable to the reaction amplitude for pickup and stripping reactions, the development of two potential forms in Chapter III provided the appropriate

starting point for the utilization of the spectator expansion. In this case the operators being expanded are the wave operators which distort the incident and exit stationary states. The spectator expansion maintains the appropriate counting. The residual target nucleons become the spectators. The eikonal approximation to the nucleon-nucleon phase shifts reduced the wave operators to a factorizable expansion in the spectator nucleons. The generalization of the eikonal amplitude to direct reaction provided a model for testing various sensitivities of the cross section.

For intermediate energies (50 MeV $\sim$ 1. GeV) we have seen that neglecting the D-state contribution to the (p,d) reaction amplitude is not suitable. At about 500 MeV the cross section arises almost purely from the D-state. Thus for a realistic treatment of the (p,d) cross section it is necessary to include the D-state contribution.

It has been demonstrated that for qualitative features and order of magnitude effects the zero-range truncation is reasonable. For detailed information it is necessary to include finite range effects in the amplitude. It may be conjectured that the simplicity inherent in ZRT may be still maintained by including the first order correction without adversely affecting the magnitude of computation unduly.

The sensitivity to density for light nuclear targets at least appears to be mainly associated with a characteristic size about which a Gaussian density or a realistic density are effectively identical. Since we have seen that the effects of nucleon-nucleon correlations are

substantial, this result is important. It may be feasible to treat comparatively high order correlation effects exactly when the density is reduced to single characteristic Gaussian.

It must be recalled that there are essentially no free parameters in these model calculations. The input for the eikonal phase shift functions was based upon the average behavior of the nucleon nucleon t-matrix information. The approximations contained in the treatment of the phase shift functions are a deficiency which will be removed in future work to achieve a more realistic treatment. However, this approximation is sufficient for the purpose of illustrating the relative energy dependence of the S-state and D-state components of the deuteron vertex. This approximation is also adequate for delineating the limitations of the zero range truncation when compared with the exact finite range computation of the deuteron vertex function.

Despite the limitations which have been described, the angular dependence of the cross-section is quite similar to that of DWBA calculations (96, 47, 51, 52). The drawbacks of DWBA computations include the use of many free parameters for the optical potential which are not uniquely determined by elastic (or inelastic) scattering information. Spectroscopic factors are sometimes adjusted after the fact to attain the proper relative strength for the reaction cross-section. Many versions of DWBA computer codes suffer the deficiency of using the zero-range approximation and including only the S-state contribution of the deuteron vertex. Some attempts have been made to correct these deficiencies in DWBA analyses, notably the work of Rost

and Shepard (52). In spite of these efforts, the current versions of DWBA analysis lack a systematic method for incorporating correlations except through the optical potential. Such a microscopic optical potential has been illustrated here and it produces a correlation equivalent to the multiple scattering formalism which was developed and applied (approximately) in this effort.

In the work presented here I have just touched the surface of the possible areas of exploitation. The extension of current efforts will direct toward the incorporation of a fuller treatment of the phase shift functions using a Harmonic Oscillator basis (at least for light nuclear systems).

Furthermore the use of H.O. representation simplifies if the inclusion of higher order correlations in the density and the possibility of explicitly including intermediate excited states, since all integrations may be treated analytically. A range of light nuclear targets will be examined over a spectrum of energies and (where available) comparison made with DWBA analyses of data. Other improvements that may be included in these studies are more precise parametrization of the short range nucleon nucleon correlations and of Pauli correlations. A study will be made to determine the sensitivity of the differential cross-section to the precision of these parametrizations and of the exchange effects that arise from antisymmetrization.

V. APPENDIX A: LEMMA A: THE IDENTITY EXPANSION

In this appendix I prove a lemma and one theorem with two specialized corollaries, which allows considerable flexibility in the construction of operator expansions. When applied to nucleus-nucleus operator expansions, this lemma is seen to be a generalization of one proven by Siciliano and Thaler. It reduces to their result in the particle-nucleus limit. The basic building blocks are purely algebraic expressions having no commutativity or inversion restrictions.

Given an arbitrary set of quantities $\{\phi^{(\nu)}_{(\mu)}\}$, where (ν) A, and (μ) B, with A > 0, B > 0, the finite series expressed by

$$\theta = \sum_{i=1}^{A} \sum_{\alpha=1}^{B} \phi_\alpha^i + \sum_{i<j}^{A} \sum_{\alpha=1}^{B} [\phi_\alpha^{ij} - \phi_\alpha^i - \phi_\alpha^j]$$

$$+ \sum_{i=1}^{A} \sum_{\alpha<\beta}^{B} [\phi_{\alpha\beta}^i - \phi_\alpha^i - \phi_\beta^i]$$

$$+ \sum_{i<j}^{A} \sum_{\alpha<\beta}^{B} [\phi_{\alpha\beta}^{ij} - \phi_\alpha^{ij} - \phi_\beta^{ij} - \phi_{\alpha\beta}^i - \phi_{\alpha\beta}^j + \phi_\alpha^i + \phi_\alpha^j + \phi_\beta^i + \phi_\beta^j]$$

$$+ \sum_{i<j<k}^{A} \sum_{\alpha=1}^{B} [\phi_\alpha^{ijk} - \phi_\alpha^{ij} - \phi_\alpha^{ik} - \phi_\alpha^{jk} + \phi_\alpha^i + \phi_\alpha^j + \phi_\alpha^k]$$

$$+ \sum_{i=1}^{A} \sum_{\alpha<\beta<\gamma}^{B} [\phi_{\alpha\beta\gamma}^i - \phi_{\alpha\beta}^i - \phi_{\alpha\gamma}^i - \phi_{\beta\gamma}^i + \phi_\alpha^i + \phi_\beta^i + \phi_\gamma^i]$$

$$+ \sum_{i<j<k}^{A} \sum_{\alpha<\beta}^{B} [\phi_{\alpha\beta}^{ijk} - \phi_{\alpha\beta}^{ij} - \phi_{\alpha\beta}^{ik} - \phi_{\alpha\beta}^{jk} - \phi_\alpha^{ijk} - \phi_\beta^{ijk}$$

$$+ \phi_\alpha^{ij} + \phi_\alpha^{ik} + \phi_\alpha^{jk} + \phi_\beta^{ij} + \phi_\beta^{ik} + \phi_\beta^{jk}$$

$$- \phi_\alpha^{i} - \phi_\beta^{i} - \phi_\alpha^{j} - \phi_\beta^{j} - \phi_\alpha^{k} - \phi_\beta^{k}]$$

$$+ \cdots$$

$$+ [\phi_{12\cdots B}^{12\cdots A} - \cdots] \qquad \text{(A.1)}$$

is exact for all arbitrary $\{\phi_{(\mu)}^{(\nu)}\}$, provided only that

$$\theta \equiv \phi_{12\cdots B}^{12\cdots A} = \phi_{(B)}^{(A)} \quad . \qquad \text{(A.2)}$$

PROOF:

The proof follows the same procedure ST used in demonstrating a more restricted version of this Lemma. We rewrite (A.1) as

$$\theta = \lim_{x\to 1} \{ \sum_{i,\alpha}^{A,B} \phi_\alpha^i + \sum_{i<j,\alpha}^{A,B} [\phi_\alpha^{ij} - x\phi_\alpha^i - x\phi_\alpha^j]$$

$$+ \sum_{i,\alpha<\beta}^{A,B} [\phi_{\alpha\beta}^i - x\phi_\alpha^i - x\phi_\beta^i]$$

$$+ \sum_{i<j,\alpha<\beta}^{A,B} [\phi_{\alpha\beta}^{ij} - x\phi_\alpha^{ij} - x\phi_\beta^{ij} - x\phi_{\alpha\beta}^i - x\phi_{\alpha\beta}^j + x^2\phi_\alpha^i + x^2\phi_\alpha^j + x^2\phi_\beta^i + x^2\phi_\beta^j]$$

$$+ \cdots\cdots\} \quad . \qquad \text{(A.3)}$$

In order to facilitate conciseness of expression and abbreviate the labor involved in demonstrating various properties of the proof, we will use the following summation convention.

$\sum$ [algebraic expression, containing the ordered superscripts

$\{(i_1, i_2, i_3, \cdots i_j),\ j<A\}$ and the ordered subscripts

$\{(\alpha_1, \alpha_2, \cdots \alpha_\beta),\ \beta<B\}$] = $\underset{i_1,<i_2<\cdots<i_j}{\overset{A}{\sum}} \quad \underset{\alpha_1<\alpha_2<\cdots<\alpha_\beta}{\overset{B}{\sum}}$

[algebraic expression].

Thus, after performing certain summations, we obtain from (A.1)

$$\theta = \lim_{x\to 1} \Bigg[[\sum \phi_\alpha^i]$$

$$+ \{[\sum \phi_\alpha^{ij}] - x(A-1)\ [\sum \phi_\alpha^i]\}$$

$$+ \{[\sum \phi_{\alpha\beta}^i] - x\ (B-1)\ [\sum \phi_\alpha^i]\}$$

$$+ \{[\sum \phi_{\alpha\beta}^{ij}] - x(B-1)\ [\sum \phi_\alpha^{ij}] - x(A-1)\ [\sum \phi_{\alpha\beta}^i] + x^2(A-1)(B-1)\ [\sum \phi_\alpha^i]\}$$

$$+ \{[\sum \phi_\alpha^{ijk}] - x\ (A-2)\ [\sum \phi_\alpha^{ij}] + x^2\ \frac{(A-1)(A-2)}{2}\ [\sum \phi_\alpha^i]\}$$

$$+ \{[\sum \phi_{\alpha\beta\gamma}^i] - x\ (B-2)\ [\sum \phi_{\alpha\beta}^i] + x^2\ \frac{(B-1)(B-2)}{2}\ [\sum \phi_\alpha^i]\}$$

$$+ \{[\sum \phi_{\alpha\beta}^{ijk}] - x\ (A-2)\ [\sum \phi_{\alpha\beta}^{ij}] - x\ (B-1)\ [\sum \phi_\alpha^{ijk}]$$

$$+ x^2\ \frac{(A-2)(A-1)}{2}\ [\sum \phi_{\alpha\beta}^i] + x^2\ (B-1)(A-2)\ [\sum \phi_\alpha^{ij}]$$

$$- x^3\ \frac{(A-1)(A-2)}{2}\ (B-1)\ [\sum \phi_\alpha^i]\}$$

$$+\cdots\cdot$$

$$+ \{\phi_{12\cdots B}^{12\cdots A} - x \cdots\ \} \Bigg] \qquad \text{(A.4)}$$

We now regroup the terms of A.4 in the following manner

$$\theta = \lim_{x\to 1}\Big[\Big[\sum \phi_\alpha^i\Big]\{1-(A-1)x-(B-1)x+(A-1)(B-1)x^2$$

$$+\frac{(A-1)(A-2)x^2}{2}+\frac{(B-1)(B-2)x^2}{2}$$

$$-\frac{(A-1)(A-2)}{2}(B-1)x^3-\frac{(A-1)(B-1)(B-2)x^3}{2}$$

$$+\frac{(A-1)(A-2)}{2}\frac{(B-1)(B-2)}{2}x^4+\cdots\}$$

$$+\Big[\sum \phi_\alpha^{ij}\Big]\{1-(B-1)x-(A-2)x+(B-1)(A-2)x^2+\frac{(B-1)(B-2)x^2}{2}+\cdots\}$$

$$+\Big[\sum \phi_{\alpha\beta}^{i}\Big]\{1-(A-1)x-(B-2)x+(A-1)(B-2)x^2+\frac{(A-1)(A-2)x^2}{2}+\cdots\}$$

$$+\Big[\sum \phi_{\alpha\beta}^{ij}\Big]\{1-(A-2)x-(B-2)x+\frac{(A-2)(A-3)x^2}{2}+\cdots\}$$

$$+\cdots\cdots$$

$$+\Big[\phi_{12\cdots B}^{12\cdots A}\Big]\Big] . \qquad \text{(A.5)}$$

And this becomes

$$\theta = \lim_{x\to 1}\Big[\Big[\sum \phi_\alpha^i\Big](1-x)^{A-1}(1-x)^{B-1}$$

$$+\Big[\sum \phi_\alpha^{ij}\Big](1-x)^{A-2}(1-x)^{B-1}$$

$$+\Big[\sum \phi_{\alpha\beta}^{i}\Big](1-x)^{A-1}(1-x)^{B-2}$$

$$+\Big[\sum \phi_{\alpha\beta}^{ij}\Big](1-x)^{A-2}(1-x)^{B-2}$$

$$+ \cdots\cdot$$

$$+ [\phi^{12\cdots A}_{12\cdots B}] \quad . \tag{A.6}$$

Taking the limit $x \to 1$, then yields

$$\theta \equiv \phi^{12\cdots A}_{12\cdots B} = \phi^{(A)}_{(B)} \quad . \tag{A.7}$$

Q.E.D.

Having shown that Lemma is true, we may now demonstrate a theorem on functional forms, and specify two useful corollaries.

Theorem A: The Functional Identity

Let X and Y have sets $\{X^{(\nu)}_{(\mu)}\}$ and $\{Y^{(\nu)}_{(\mu)}\}$ respectively which satisfy Lemma A. Then for Z defined by

$$Z \equiv F(X,Y) \tag{A.8}$$

if

$$f^{(A)}_{(B)} \equiv F(X^{(A)}_{(B)};\ Y^{(A)}_{(B)}) \tag{A.9}$$

the set $\{Z^{(\nu)}_{(\mu)}\}$ described by the otherwise arbitrary set of functionals $\{f^{(\nu)}_{(\mu)}\}$ as follows

$$Z^{(\nu)}_{(\mu)} \equiv f^{(\nu)}_{(\mu)}\ (\{X^{(\nu')}_{(\mu')}\};\ \{Y^{(\nu')}_{(\mu')}\}) \tag{A.10}$$

satisfies Lemma A for Z.

PROOF:

(i) $\quad X^{(A)}_{(B)} \equiv X;\ Y^{(A)}_{(B)} \equiv Y \qquad$ by Lemma A

(ii) $f^{(A)}_{(B)} = F(X,Y)$ by (A.9) and (i)

(iii) $Z^{(A)}_{(B)} = F(X,Y)$ by (A.10) and (ii)

(iv) $Z^{(A)}_{(B)} = Z$ by (A.8) and (iii) .

Therefore the condition necessary for the set $\{Z^{(\nu)}_{(\mu)}\}$ to satisfy Lemma A is met.

Q.E.D.

We can now specify Theorem A to yield two corollaries.

Corollary I: Addition Corollary

Given X and Y, having sets $\{X^{(\nu)}_{(\mu)}\}$ and $\{Y^{(\nu)}_{(\mu)}\}$ respectively, which satisfy Lemma A, then for

$$Z \equiv X + Y \quad , \tag{A.11}$$

the set $\{Z^{(\nu)}_{(\mu)}\}$, defined by

$$Z^{(\nu)}_{(\mu)} \equiv X^{(\nu)}_{(\mu)} + Y^{(\nu)}_{(\mu)} \tag{A.12}$$

also satisfies Lemma A.

Corollary II: Multiplication Corollary

Given X and Y, having sets $\{X^{(\nu)}_{(\mu)}\}$ and $\{Y^{(\nu)}_{(\mu)}\}$ respectively, satisfying Lemma A, then for

$$Z = XY \tag{A.13}$$

the set $\{Z^{(\nu)}_{(\mu)}\}$, defined by

$$Z^{(\nu)}_{(\mu)} \equiv X^{(\nu)}_{(\mu)}\, Y^{(\nu)}_{(\mu)} \tag{A.14}$$

also satisfies Lemma A.

It is convenient at this juncture to point out that Corollaries I and II are more fundamental than their derivation implies. One can show trivially for Corollary I that X + Y in terms of the expansion factors yields identically the terms given by the definition (A.12). Furthermore, while it is not as obvious, the expansion of Z = XY in terms of $\{X^{(\nu)}_{(\mu)}\}$ and $\{Y^{(\nu)}_{(\mu)}\}$, permits a rearrangement which yields the identical terms of definition (A.14). In this regard, we must note that Theorem A provides a minimal condition upon the expansion factors of Z; whereas having defined the expansions of X and Y we obtain a specific expansion of Z in terms of the sets $\{X^{(\nu)}_{(\mu)}\}$ and $\{Y^{(\nu)}_{(\mu)}\}$. This facet of Corollaries I and II removes a degree of the arbitrariness inherent in such expansions. Physically, this implies that an operator expression has a 'natural' set of expansion factors whose form will be constrained by the form of the operator expression. For example, Z = XY, has the 'natural' set of $\{Z^{(\nu)}_{(\mu)} = X^{(\nu)}_{(\mu)}\, Y^{(\nu)}_{(\mu)}\}$, in which the arbitrariness has been restricted to that of the X and Y expansion sets.

VI. APPENDIX B: PROJECTION OPERATOR IDENTITIES

Consider a set of m operators $\{P_i\}^m$ defined in the N-body space V(N). By definition the $\{P_i\}$ are idempotent projection operators in V(N), provided that the following conditions are satisfied for all P_i in the set

$$\sum_{i=1}^{m} P_i = 1(N) \tag{B.1}$$

$$P_i P_j = \delta_{ij} P_i \quad . \tag{B.2}$$

Take an operator expression (B.3) defined in V(N), where

$$A = B + BCA \quad . \tag{B.3}$$

A, B, and C are unspecified N-body operators. It is possible to generate an equivalent set of operator expressions for (B.3) by using (B.1). Let m = 2, and $P_1 = P$ and $P_2 = Q$. We may now rewrite (B.3) as follows

$$A = B + BCA = B + BC(P + Q)A \quad ,$$

$$A = B + BCQA + BCPA \quad . \tag{B.4}$$

Inserting the expression (B.4) for A into the second term of (B.4) yields

$$A = B + BCQB + BCQBCQA + BCPA + BCQBCPA \quad . \tag{B.5}$$

And again using (B.4) we obtain

$$A = B + BCQB + BCQBCQB + BCQBCQBCQA + BCPA + BCQBCPA + BCQBCQBCPA \quad . \tag{B.6}$$

We see that (B.6) may more concisely be written as

$$A = B\left[\sum_{n=0}^{2}(CQB)^n\right] + (BCQ)^3A + B\left[\sum_{n=0}^{2}(CQB)^n\right]CPA \quad . \qquad (B.7)$$

Iteration of this procedure yields

$$A = B\left[\sum_{n=0}^{\infty}(CQB)^n\right] + B\left[\sum_{n=0}^{\infty}(CQB)^n\right]CPA \quad . \qquad (B.8)$$

We may identify a new operator D defined by the expression

$$D \equiv B\left[\sum_{n=0}^{\infty}(CQB)^n\right] \quad . \qquad (B.9)$$

An alternate form for D is seen to be

$$D = B + BCQD \quad . \qquad (B.10)$$

Having defined D we now rewrite (B.8) in a much simpler form as

$$A = D + DCPA \quad . \qquad (B.11)$$

By using the projection operators P and Q we have separated the single operator expression (B.3) into two expressions (B.10) and (B.11) having the same content.

Using the same procedures it is possible to obtain three expressions for the set of three projectors P, Q_1, Q_2. Let $Q = Q_1 + Q_2$. Replacing Q in (B.10) then yields after some manipulation.

$$E = B + BDQ_1E$$

$$D = E + ECQ_2D \quad .$$

We can generalize this procedure to m projectors. The result is a set of m-equations containing various portions of the full content of (B.3). The generalized set looks like

$$D_{m-\ell} = D_{m-\ell+1} + D_{m-\ell+1}\, C\, P_{m-\ell}\, D_{m-\ell} \tag{B.12}$$

where $\ell = 0, 1, 2. \cdots (M - 1)$, and $D_{m+1} = B$ and $D_1 = A$.

VII. APPENDIX C: DEUTERON VERTEX FUNCTIONS

A modern realistic nucleon-nucleon interaction V_{NN} contains at least spin, isospin, spin-orbit and tensor coupling terms. Such couplings even in the presence of a soft core render the utilization of these interactions in the formation of transition amplitudes extremely cumbersome. For general usage in reactions, it is desirable to obtain from V_{NN} and its corresponding deuteron wavefunction ϕ_{dtn} a quantity which is approximately "invariant". That is, essentially contains all the deuteron properties, but may be regarded as comparatively insensitive to the detailed features of the interaction. These functions I shall call the deuteron vertex function.

Consider the Reid soft core interaction V_{Reid} and wavefunctions ϕ_{Reid} (56). For the standard deuteron, the $(^3S_1 - {}^3D_1)$ configuration, the potential is defined as

$$V_{NN} = V_C + V_T\, S_{12} + V_{LS}\, L \cdot S \quad , \tag{C.1}$$

where

$$V_C = - h\, e^{-x}/x + 105.468\, e^{-2x}/x - 3187.8\, e^{-4x}/x + 9924.3\, e^{-6x}/x \tag{C.2}$$

$$V_T = - h(1 + 3/x + 3/x^2)e^{-x} - (12/x + 3/x^2)e^{-4x}/x + 351.77\, e^{-4x} - 1673.5\, e^{-6x}/x \tag{C.3}$$

$$V_{LS} = 708.91\, e^{-4x}/x - 2713\, e^{-6x}/x \tag{C.4}$$

and

$$x = (0.7\ F^{-1})r \qquad h = 10.463\ \text{MeV} \quad .$$

The deuteron wavefunction for the ($^3S_1$ - $^3D_1$) configuration is written as

$$\phi_{dtn}(1,M_J) = \frac{1}{r}[X_{1M_J}\ Y_{00}\ U_S(r) + \sum_{m_D\mu_S} \langle 1M_J|2m_2\ 1\mu_S\rangle X_{1\mu_S}\ Y_{2m_D}(r)W_D(r)] \quad . \tag{C.5}$$

The deuteron vertex functions $d_{\ell m_\ell}$ are defined as the angular momentum decoupled product of $D_{Jm_j} = V_{NN}\ \phi_{dtn}$. That is, for the $^3S_1$ - $^3D_1$ configuration

$$D_{1m_J}(\vec{r},\tilde{\sigma}) = X_{1m_J}\ Y_{00}\ (\frac{1}{r}\ U_S^{eff}(r)) + \sum_{m_D,\mu_S} \langle 1m_J|2m_D\ 1\mu_S\rangle$$

$$x\ X_{1\mu_S}\ Y_{2m_D}(r)(\frac{1}{r}\ U_D^{eff}(r)) \quad , \tag{C.6}$$

where

$$U_S^{eff}(r) = V_C(r)\ U_S(r) + \sqrt{8}\ V_T(r)\ W_D(r) \tag{C.7}$$

and

$$U_D^{eff} = (V_C(r) - 2V_T(r) - 3V_{LS}(r))W_D(r) + \sqrt{8}\ V_T(r)\ U_S(r) \quad . \tag{C.8}$$

An alternative expression for Eq. (C.6) in terms of the vertex functions is

$$D_{1m_S} = X_{1m_J}\ d_{00}(\vec{r}) + \sum \langle 1m_J|2m_D\ 1\mu_S\rangle X_{1\mu_S}\ d_{2m_D}(\vec{r}) \quad , \tag{C.9}$$

where

$$d_{00}(\vec{r}) = Y_{00} \frac{1}{r} U_S^{eff}(r) \tag{C.10}$$

and

$$d_{2m_D}(\vec{r}) = Y_{2m_D}(\hat{r}) \frac{1}{r} U_D^{eff}(r) \quad . \tag{C.11}$$

The vertex functions $d_{\ell m_\ell}(\vec{r})$ are simpler to work with than is the deuteron potential V_{NN}.

While the interaction V_{NN} is known analytically, the corresponding wavefunctions are created on a numerical mesh. Thus the function $U_L^{eff}(r)$ are known numerically, rather than analytically. It is convenient to parametrize these functions in the following format

$$U_L^{eff}(r) \doteq r^{L+1} \sum_{i=1}^{M} C_i^L e^{-\alpha_i^L r^2} \quad . \tag{C.12}$$

The values of the C_i^L and α_i^L are given for the $^3S_1 - {}^3D_1$ configuration in the tables.

VIII. APPENDIX D: ANALYTIC EXPRESSION FOR FOLDING SPHERICAL HARMONICS IN AN ASYMMETRIC GAUSSIAN FIELD

A. General Formalism

The harmonic folded integral in a deformed Gaussian field is defined as

$$I(Q{:}LM;\ P{:}\ell m) \equiv \int e^{i\vec{Q}\cdot\vec{R}}(R)^L\ Y^*_{LM}(\hat{R})\ e^{i\vec{P}\cdot\vec{r}}(r)^\ell\ Y_{\ell m}(r)\ E(\vec{R},\vec{r})d^3R\ d^3r \quad , \tag{D.1}$$

where

$$E(R,r) \equiv \exp\{-A_xX^2 - A_yY^2 - A_zZ^2 + 2B_xxX + 2B_yyY + 2B_zzZ - a_xx^2 - a_yy^2 - a_zz^2\} \quad . \tag{D.2}$$

In Eq. (D.1) the $Y_{LM}(\hat{R})$ are the standard spherical harmonics, obeying the normalization convention

$$\int Y^*_{LM}(\hat{R})\ Y_{L'M'}(\hat{R})\ d\Omega_R = \delta_{LL'}\ \delta_{MM'} \quad , \tag{D.3}$$

and satisfying the identity

$$Y_{LM}(\hat{R}) = (-)^M\ Y_{L(-M)}(\hat{R}) \quad . \tag{D.4}$$

For arbitrary angular momentum states (LM) and (ℓm), the integral of Eq. (D.1) is nontractable. Consequently it is desirable to replace it by an equivalent representation in which the angular momentum dependence does not appear explicitly in the integral, but is rather subsumed in a set of differential operators.

A general polynomial differential operator of rank ℓ and order m, denoted by $D_{m}(q)$, may be defined in the following form

$$D_{\ell m}(\vec{q}) \equiv \sum_{\alpha+\beta+\gamma=\ell} a^{\ell m}_{\alpha\beta\gamma} (\partial/\partial q_x)^{\alpha}(\partial/\partial q_y)^{\beta}(\partial/\partial q_z)^{\gamma} \quad . \tag{D.5}$$

By requiring the $D_{\ell m}$ to satisfy the following expressions

$$D_{\ell m}(\vec{q})e^{i\vec{q}\cdot\vec{r}} = (r)^{\ell} Y_{\ell m}(\hat{r})e^{i\vec{q}\cdot\vec{r}} \quad , \tag{D.6a}$$

and

$$D_{\ell m}(\vec{q})e^{i\vec{q}\cdot\vec{r}} = (r)^{\ell} Y^{*}_{\ell m}(\hat{r})e^{i\vec{q}\cdot\vec{r}} \quad , \tag{D.6b}$$

then for a given (ℓm), the coefficients $a^{\ell m}_{\alpha\beta\gamma}$ are completely determined. Furthermore, using Eqs. (D.4) and (D.6b), the following identity is obtained.

$$D_{\ell m}(\vec{q})^{*} = (-)^{m} D_{\ell -m}(\vec{q}) \quad . \tag{D.7}$$

For reference purposes, the spherical harmonics and operator equivalents for ℓ = 0 through ℓ = 3 are listed below. For brevity, the notation is defined such that $\partial_x = \partial/\partial q_x$, $\partial_y = \partial/\partial q_y$, and $\partial_z = \partial/\partial q_z$

ℓm	$r^{\ell} Y_{\ell m}(\hat{r})$	$D_{\ell m}(\vec{q})$	
00	$(1/4\ \pi)^{\frac{1}{2}}$	$(1/4\ \pi)^{\frac{1}{2}}$	(D.8)
10	$(3/4\ \pi)^{\frac{1}{2}}z$	$(3/4\ \pi)^{\frac{1}{2}}(-i)\ _{z}$	(D.9a)
11	$-(3/8\ \pi)^{\frac{1}{2}}be^{i\phi}$	$-(3/8\ \pi)^{\frac{1}{2}}(-i)(\partial_x+i\partial_y)$	(D.9b)

$$20 \quad (5/16\,\pi)^{\frac{1}{2}}(2z^2 - b^2) \qquad (5/16\,\pi)^{\frac{1}{2}}(-i)^2(2\partial_z^2 - \partial_x^2 - \partial_y^2) \qquad \text{(D.10a)}$$

$$21 \quad -(15/8\,\pi)^{\frac{1}{2}} zbe^{i\phi} \qquad -(15/8\,\pi)^{\frac{1}{2}}(-i)^2(\partial_z)(\partial_x + i\partial_y) \qquad \text{(D.10b)}$$

$$22 \quad (15/32\,\pi)^{\frac{1}{2}} b^2\, e^{i2\phi} \qquad (15/32\,\pi)^{\frac{1}{2}}(-i)^2(\partial_x + i\partial_y)^2 \qquad \text{(D.10c)}$$

$$30 \quad (7/16\,\pi)^{\frac{1}{2}}(2z^2 - 3b^2)z \qquad (7/16\,\pi)^{\frac{1}{2}}(-i)^3(2\partial_z^2 - 3\partial_x^2 - 3\partial_y^2)(\partial_z) \qquad \text{(D.11a)}$$

$$31 \quad -(21/64\,\pi)^{\frac{1}{2}}(4z^2 - b^2)be^{i\phi} \qquad -(21/64\,\pi)^{\frac{1}{2}}(-i)^3(4\partial_z^2 - \partial_x^2 - \partial_y^2)(\partial_x + i\partial_y) \qquad \text{(D.11b)}$$

$$32 \quad (105/32\,\pi)^{\frac{1}{2}} zb^2\, e^{i2\phi} \qquad (105/32\,\pi)^{\frac{1}{2}}(-i)^3(\partial_z)(\partial_x + i\partial_y)^2 \qquad \text{(D.11c)}$$

$$33 \quad -(35/64\,\pi)^{\frac{1}{2}} b^3\, e^{i3\phi} \qquad -(35/64\,\pi)^{\frac{1}{2}}(-i)^3(\partial_x + i\partial_y)^3 \quad . \qquad \text{(D.11d)}$$

The generalization required to generate higher rank (ℓm) values is evident from the systematics of Eqs. (D.8) through (D.11).

Using Eqs. (D.6a) and (D.6b), Eq. (D.1) may be rewritten as

$$I(\vec{Q}:LM;\ \vec{P}:\ell m) \;=\; D^*_{LM}(\vec{Q})\; D_{\ell m}(\vec{P})\; \mathcal{J}(\vec{Q},\vec{P}) \quad , \qquad \text{(D.12)}$$

where

$$\mathcal{J}(\vec{Q},\vec{P}) \;\equiv\; \int e^{i\vec{Q}\cdot\vec{R}}\; e^{i\vec{P}\cdot\vec{r}}\; E(\vec{R},\vec{r})\; d^3R\; d^3r \quad . \qquad \text{(D.13)}$$

Note that in Eq. (D.12) the angular momentum dependence has been removed from the integral of Eq. (D.1). The integral $\mathcal{J}(\vec{Q},\vec{P})$ is the double Fourier transform of the coupled asymmetric Gaussian form factor

$E(\vec{R},\vec{r})$. Evaluating Eq. (D.13) yields

$$\mathcal{I}(\vec{Q},\vec{P}) = \pi^3 I_x(Q_x,P_x) I_y(Q_y,P_y) I_z(Q_z,P_z) \tag{D.14}$$

where

$$I_j(Q_j,P_j) = \left(\frac{1}{A_j a_j - B_j^2}\right)^{1/2} \exp\left[-\frac{1}{4(A_j a_j - B_j^2}\,(Q_j^2 a_j + P_j^2 A_j + 2P_j Q_j B_j)\right] \tag{D.15}$$

and j may be x, y or z.

Thus the harmonic folded integral $I(\vec{Q}:LM;\ \vec{P}:\ell m)$ has been reduced to a general analytic expression in which the differential operators $D_{\ell m}$ and D_{LM} generate the desired angular momentum coupling.

B. Specific Application

Consider the situation in which the asymmetry of the Gaussian $E(\vec{R},\vec{r})$ has the form

$$E(\vec{R},\vec{r}) = \exp(-A_z Z^2 - a_z z^2 - A_p B^2 - a_p b^2 + 2d_p \vec{b}\cdot\vec{B}) \quad . \tag{D.16}$$

That is, $B_z = b_z = 0$, $A_x = A_y = A_p$, $a_x = a_y = a_p$, and $B_x = B_y = d_p$. That is, there is a symmetry coupling in the azimuthal direction. Making use of Eqs. (D.14) and (D.15), and the special case of $E(\vec{R},\vec{r})$ represented by (D.16) it is possible to obtain for $\mathcal{I}(\vec{Q},\vec{P})$ the form

$$\mathcal{I}(\vec{Q},\vec{P}) = \pi^3/(A_z a_z)^{\frac{1}{2}}/(A_p a_p - d_p^2)\exp(-Q_z^2/4A_z - P_z^2/4a_z)$$

$$\times \exp\left\{-\frac{Q_p^2}{4}\left(\frac{a_p}{A_p a_p - d_p^2}\right) - \frac{P_p^2}{4}\left(\frac{A_p}{A_p a_p - d_p^2}\right) - 2\,\frac{\vec{Q}_p \cdot \vec{P}_p}{4}\left(\frac{d_p}{A_p a_p - d_p^2}\right)\right\} \tag{D.17}$$

Here $Q_p^2 = Q_x^2 + Q_y^2$ and $P_p^2 = P_x^2 + P_y^2$.

In order to expedite the remaining manipulation, the following definitions shall be used to simplify Eq. (D.17)

$$C = \pi^3/(A_z a_z)^{\frac{1}{2}}/(A_p a_p - d_p^2) \tag{D.18a}$$

$$\alpha = 1/A_z \tag{D.18b}$$

$$\beta = 1/a_z \tag{D.18c}$$

$$\gamma = a_p/(A_p a_p - d_p^2) \tag{D.18d}$$

$$\delta = A_p/(A_p a_p - d_p^2) \tag{D.18e}$$

$$\varepsilon = d_p/(A_p a_p - d_p^2) \tag{D.18f}$$

$$Q_p\, e^{\pm i\phi'} = Q_x \pm iQ_y \tag{D.18g}$$

$$P_p\, e^{\pm i\phi} = P_x \pm iP_y \quad . \tag{D.18h}$$

Making use of Eqs. (D.18a-h), Eq. (D.17) may be rewritten as

$$\mathcal{J}(\vec{Q},\vec{P}) = C \exp\left\{-\alpha\,\frac{Q_z^2}{4} - \beta\,\frac{P_z^2}{4} - \gamma\,\frac{Q_p^2}{4} - \delta\,\frac{P_p^2}{4} - \varepsilon\,2\,\frac{P_p \cdot Q_p}{4}\right\} \quad . \tag{D.19}$$

This leads to a simple evaluation of the harmonic integral

$$I(\vec{Q}:LM;\ \vec{P}:\ell m) = D^{*}_{LM}(\vec{Q})\ D_{\ell m}(\vec{P})\ \vartheta(\vec{Q},\vec{P}) \quad , \tag{D.20a}$$

$$= (-)^{M}\ D_{L(-M)}(\vec{Q})\ D_{\ell m}(\vec{P})\ \vartheta(\vec{Q},\vec{P}) \quad . \tag{D.20b}$$

Using the representations of $D_{LM}(\vec{q})$ given by Eqs. (D.8) through (D.10), it is possible to generate the following (LM), (ℓm) couplings for reference.

$$I(\vec{Q}:00;\ \vec{P}:00) = 1/4\ \pi\ \vartheta(\vec{Q},\vec{P}) \tag{D.21}$$

$$I(\vec{Q}:00;\ \vec{P}:1m) = \left(\frac{i}{2}\right)\frac{1}{\sqrt{4\pi}}\vartheta(\vec{Q},\vec{P})\ x \begin{cases} \sqrt{3/4\ \pi}\ \beta P_z & m=0 \\ \mp\sqrt{3/8\ \pi}\ (\delta P_p\, e^{\pm i\phi} + \varepsilon Q_p\, e^{\pm i\phi'}) & m=\pm 1 \end{cases} \tag{D.22}$$

$$I(\vec{Q}:00;\ \vec{P}:2m) = \left(\frac{i}{2}\right)^{2}\frac{1}{\sqrt{4\pi}}\vartheta(\vec{Q},\vec{P})\ x$$

$$\begin{cases} \sqrt{5/16\ \pi}\{2(\beta^2 P_z^2 - 2\beta) - (\delta^2 P_p^2 + \varepsilon^2 Q_p^2 + 2\delta\varepsilon Q_p \cdot P_p - 4\delta) & m=0 \\ \mp\sqrt{15/8\ \pi}\,(\beta P_z)(\delta P_p\, e^{\pm i\phi} + \varepsilon Q_p\, e^{\pm i\phi'}) & m=\pm 1 \\ \sqrt{15/32\ \pi}\,(\delta P_p\, e^{\pm i\phi} + \varepsilon Q_p\, e^{\pm i\phi'})^2 & m=\pm 2 \end{cases} \tag{D.23}$$

$$I(\vec{Q}:00;\ \vec{P}:3m) = \left(\frac{i}{2}\right)^{3}\frac{1}{\sqrt{4\pi}}\vartheta(\vec{Q},\vec{P})\ x$$

$$\sqrt{7/16\,\pi}\,[\{2(\beta^2P_z^2-\beta)-3(\delta^2P_p^2+\varepsilon^2Q_p^2+2\delta\varepsilon\vec{P}_p\cdot\vec{Q}_p-4)\}(\beta P_z)-8\beta^2P_z] \quad m=0$$

$$\begin{cases} \sqrt{21/6\,\pi}\,[\{4(\beta^2P_z^2-\beta)-(\delta^2P_p^2+\varepsilon^2Q_p^2+2\delta\varepsilon\vec{P}_p\cdot\vec{Q}_p)\}(\delta P_p\,e^{\pm i\phi}+\varepsilon Q_p\,e^{\pm i\phi'}] & m=\pm1 \\ \sqrt{105/32\,\pi}\,(\beta P_z)(\delta P_p\,e^{\pm i\phi}+\varepsilon Q_p\,e^{\pm i\phi'})^2 & m=\pm2 \\ \sqrt{35/64\,\pi}\,(\delta P_p\,e^{\pm i\phi}+\varepsilon Q_p\,e^{\pm i\phi'})^3 & m=\pm3 \end{cases}$$

(D.24)

$$I(\vec{Q}:2M;\vec{P}:00) = \left(\frac{i}{2}\right)^2\frac{1}{\sqrt{4\pi}}\,\mathcal{J}(\vec{Q},\vec{P})\ x$$

$$\begin{cases} \sqrt{5/16\,\pi}\,\{2(\alpha^2Q_z^2-2\alpha)-(\gamma^2Q_p^2+\varepsilon^2P_p^2+2\gamma\varepsilon\vec{Q}_p\cdot\vec{P}_p-4\gamma)\} & M=0 \\ \sqrt{15/8\,\pi}\,(\alpha Q_z)(\gamma Q_p\,e^{\mp i\phi}+\varepsilon P_p\,e^{\mp i\phi'}) & M=\pm1 \\ \sqrt{15/32\,\pi}\,(\gamma Q_p\,e^{\mp i\phi}+\varepsilon P_p\,e^{\mp i\phi'})^2 & M=\pm2 \end{cases}$$

(D.25)

$$I(\vec{Q}:2M;\ \vec{P}:1m) = \left(\frac{i}{2}\right)^3\mathcal{J}(\vec{Q},\vec{P})\ x$$

$$\begin{cases} \sqrt{5/16\pi}\ \sqrt{3/4\pi}\ \{2(\alpha^2Q_z^2-2\alpha)-(\gamma^2Q_p^2+\varepsilon^2P_p^2+2\gamma\varepsilon\vec{Q}_p\cdot\vec{P}_p-4\gamma)\}\beta P_z & M=0 \quad m=0 \\ \mp\sqrt{5/16\pi}\ \sqrt{3/8\pi}\ [\{2(\alpha^2Q_z^2-2\alpha)-(\gamma^2Q_p^2+\varepsilon^2P_p^2+2\gamma\varepsilon\vec{Q}_p\cdot\vec{P}_p-4\gamma)\ x \\ \qquad (\delta P_p\,e^{\pm i\phi}+\varepsilon Q_p\,e^{\pm i\phi'})+4\varepsilon(\varepsilon P_p\,e^{\pm i\phi}+\gamma Q_p\,e^{\pm i\phi'})] & M=0 \quad m=\pm1 \end{cases}$$

$$\left.\begin{array}{l}
\pm \sqrt{15/8\pi}\ \sqrt{3/4\pi}\ (\alpha Q_z)(\gamma Q_p\ e^{\mp i\phi} + \varepsilon P_p\ e^{\mp i\phi'})(\beta P_z) \qquad M=\pm 1 \quad m=0 \\
\\
-\sqrt{15/8\pi}\ \sqrt{3/8\pi}\ [(\alpha Q_z)(\gamma Q_p\ e^{\mp i\phi} + \varepsilon P_p\ e^{\mp i\phi'})(\delta P_p\ e^{\pm i\phi} + \varepsilon Q_p\ e^{\pm i\phi'}) \\
\\
\qquad - 4\varepsilon\alpha Q_z \qquad M=+1 \quad m=+1 \\
\qquad\qquad \text{or} \\
\qquad\qquad M=-1 \quad m=-1 \\
\\
\sqrt{15/8\pi}\ \sqrt{3/8\pi}\ [(\alpha Q_z)(\gamma Q_p\ e^{\pm i\phi} + \varepsilon P_p\ e^{\pm i\phi'})(\delta P_p\ e^{\pm i\phi} + \varepsilon Q_p\ e^{\pm i\phi'})] \\
\\
\qquad\qquad M=-1 \quad m=+1 \\
\qquad\qquad \text{or} \\
\qquad\qquad M=+1 \quad m=-1 \\
\\
\sqrt{15/32\pi}\ \sqrt{3/4\pi}\ (\gamma Q_p\ e^{\mp i\phi} + \varepsilon P_p\ e^{\mp i\phi'})^2(\beta P_z) \qquad M=\pm 2 \quad m=0 \\
\\
\mp \sqrt{15/32\pi}\ \sqrt{3/8\pi}\ [(\gamma Q_p\ e^{\mp i\phi} + \varepsilon P_p\ e^{\mp i\phi'})^2(\delta P_p\ e^{\pm i\phi} + \varepsilon Q_p\ e^{\pm i\phi'}) \\
\\
\qquad - 4\varepsilon(\gamma Q_p\ e^{\mp i\phi} + \varepsilon P_p\ e^{\mp i\phi'})] \qquad M=+2 \quad m=+1 \\
\qquad\qquad \text{or} \\
\qquad\qquad M=-2 \quad m=-1 \\
\\
\mp \sqrt{15/32\pi}\ \sqrt{3/8\pi}\ [(\gamma Q_p\ e^{\pm i\phi} + \varepsilon P_p\ e^{\pm i\phi'})^2(\delta P_p\ e^{\pm i\phi} + \varepsilon Q_p\ e^{\pm i\phi'})] \\
\\
\qquad\qquad M=-2 \quad m=+1 \\
\qquad\qquad \text{or} \\
\qquad\qquad M=+2 \quad m=-1
\end{array}\right\} \qquad (D.26)$$

IX. REFERENCES

1. L. D. Faddeev, Soviet Phys. - JETP 12, 1014 (1961).
2. L. D. Faddeev, in Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory (Israel Program for Scientific Translation, Jerusalem, 1965, distributed by Oldbourne Press, London).
3. S. Weinberg, Phys. Rev. 130, 776 (1963).
4. S. Weinberg, Phys. Rev. 131, 440 (1963).
5. M. Scadron and S. Weinberg, Phys. Rev. 133, B1589 (1964).
6. C. Lovelace, Phys. Rev. 135, B1225 (1964).
7. D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cim. 43A, 258 (1966).
8. P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967).
9. E. O. Alt, P. Grassberger and W. Sandhas, Nucl. Phys. B2, 167 (1967).
10. E. F. Redish, Nucl. Phys. A235, 82 (1974).
11. D. J. Kouri and F. S. Levin, Phys. Rev. C 16, 556 (1977).
12. Gy. Bencze and P. C. Tandy, Phys. Rev. C 16, 564 (1977).
13. K. L. Kowalski, Phys. Rev. C 16, 1735 (1977).
14. K. M. Watson, Phys. Rev. 89, 575 (1953).
15. R. J. Glauber, in Lectures in Theoretical Physics, Vol. I (Interscience, New York, 1959).
16. A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. 8, 551 (1959).
17. O. Kofoed-Hansen, Nuovo Cim. 60A, 621 (1968).
18. J. Formanek, Nucl. Phys. B12, 441 (1969).
19. S. J. Wallace, Ann. Phys. 78, 190 (1973).
20. S. J. Wallace, Phys. Rev. C 8, 2043 (1973).
21. V. Franko, Phys. Rev. 175, 1376 (1968).

22. C. W. Wong and S. K. Young, Phys. Rev. C 12, 1301 (1975).

23. D. J. Ernst, J. T. Londergan, Gerald A. Miller, and R. M. Thaler, Phys. Rev. C 16, 537 (1977).

24. E. Siciliano and R. M. Thaler, Phys. Rev. C 16, 1322 (1977).

25. L. D. Ludeking and J. P. Vary, Phys. Rev. C, to be published.

26. K. L. Kowalski, Nucl. Phys., to be published.

27. J. Y. Park, W. Scheid, and W. Greiner, Phys. Rev. C 6, 1565 (1965).

28. R. M. Thaler, Physics Department, Case Western Reserve University, Cleveland, Ohio, 1979 (private communication).

29. R. Kubo, J. Phys. Soc. Japan 17, 1100 (1962).

30. M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91, 398 (1953).

31. L. I. Schiff, Phys. Rev. 103, 443 (1956).

32. G. Moliere, Z. Natur. Forsch. 2, 133 (1947).

33. N. Austern, in Selected Topics in Nuclear Theory, edited by F. Janouch (Int. Atomic Energy Agency, Vienna, 1963).

34. S. K. Penny and G. R. Satchler, Nucl. Phys. 53, 145 (1964).

35. A. Messiah, Quantum Mechanics, Vol. II (John Wiley & Sons, New York, 1966).

36. B. A. Lippmann, Phys. Rev. 102, 264 (1956).

37. L. S. Rodberg and R. M. Thaler, Quantum Theory and Scattering (Academic Press, New York, 1967).

38. B. H. Brandow, Rev. Mod. Phys. 39, 771 (1967).

39. P. J. Ellis and E. Osnes, Rev. Mod. Phys. 49, 777 (1977).

40. T. Fujita and J. Hüfner, Nucl. Phys. A314, 317 (1979).

41. H. Feshbach, Ann. Phys. 19, 287 (1962).

42. J. P. Vary and C. B. Dover, Microscopic Models for Heavy Ion Scattering at Low, Intermediate, and High Energies, Proceedings of the Second High Energy Heavy Ion Summer Study, Berkeley, 1974, LBL-3675.

43. B. J. B. Crowley and B. Buck, General Eikonal Theory of Quantal Scattering, University of Oxford, Department of Theoretical Physics, 1977 (unpublished manuscript).

44. M. L. Goldberger and K. M. Watson, Collision Theory (John Wiley & Sons, New York, 1964).

45. W. Czyz and W. Maximon, Ann. Phys. 52, 59 (1969).

46. G. J. Igo, in High Energy Physics and Nuclear Structure - 1975, AIP Conference Proceedings No. 26, edited by D. E. Nagle, A. S. Goldhaber, C. K. Hargrave, R. L. Burman, and B. G. Storms (American Institute of Physics, New York, 1975).

47. G. J. Igo, Rev. Mod. Phys. 50, 523 (1978).

48. A. Chaumeaux, V. Layly, and R. Schaeffer, Ann. Phys. 116, 247 (1978).

49. A. Tekou, Nucleon Transfer Reaction and the Eikonal Approximation, Department de Physique Nucleaire, Centre d'Etudes Nucleaires de Saclay, France (unpublished).

50. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55 (U.S. Government Printing Office, Washington, D.C., 1968).

51. S. D. Baker, R. Bertini, R. Beurtey, F. Brochard, G. Bruge, H. Catz, A. Chaumeaux, G. Cuijanovich, J. M. Durand, J. C. Faiure, J. M. Fontaine, D. Garreta, F. Hibou, D. Legrand, J. C. Lugol, J. Saudinos, and J. Thirion, Phys. Lett. B52, 57 (1974).

52. E. Rost and J. R. Shepard, Phys. Lett. B59, 413 (1975).

53. D. Vautherin and D. M. Brink, Phys. Rev. C 5, 626 (1972). We gratefully acknowledge receipt of the DDHF code from John Negele.

54. J. Yvon, in Correlations and Entropy in Classical Statistical Mechanics (Pergamon, Oxford, 1969).

55. L. L. Foldy and J. D. Walecka, Ann. Phys. 54, 447 (1969).

56. R. V. Reid, Ann. Phys. 50, 411 (1968).

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and knowledge shall be increased.

Book of Daniel
Chapter XII, Verse 4